M2 NPAC 2018/2019

QFT - part 2 **TD n°2**

Green's functions and propagators

Useful relations :

$$\int_{-\infty}^{+\infty} dy\, e^{iyz} = 2\pi\delta(z); \qquad \frac{d}{dz}\theta(z) = \delta(z)$$

Cauchy's theorem :

$$\oint dz \frac{f(z)}{z-w} = 2n\pi i f(w)$$

where n is the number of times the integration contour goes around w anti-clockwise.

1. KG equation coupled to an external source

Consider the equation for a real scalar field with an external source :

$$(\Box + m^2)\phi(\vec{x},t) = J(\vec{x},t). \tag{1}$$

1. Assuming there is an instant t_0 such that $J(\vec{x}, t < t_0) = 0$, write, in terms of an appropriate Green's function, the solution $\phi(\vec{x},t)$ which satisfies $\phi(\vec{x}, t < t_0) = 0$. Which is the correctn Green's function in this case ?
2. Find the explicit solution $\phi(\vec{x},t)$ in the particular case :

$$J(\vec{x},t) = j_0\,\theta(t)\,e^{-\mu t}, \qquad j_0, \mu > 0, \qquad \theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$

3. Recall the solution for the classical Klein-Gordon (1) in the absence of sources :

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a(\vec{k})\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + a^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t} \right) \tag{3}$$

where $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$. Show that, for $t \to \infty$, the solution found in the previous point reduces to a solution of the free equation with coefficients :

$$a(\vec{k}) = -\frac{(2\pi)^{3/2} j_0}{(m + i\mu)\sqrt{2m}}\,\delta^3(\vec{k}). \tag{4}$$

3. Green's function for Helmoltz equation

Consider the following static equation in d=3 space dimensions (no time) :

$$(-\nabla^2 + m^2)\phi(\vec{x}) = 0 \tag{5}$$

where $\phi(\vec{x})$ is a scalar field (with respect rotations in 3d). We want to construct the associated Green's function, $G(\vec{x})$, i.e. the solution of

$$(-\nabla^2 + m^2)G(\vec{x}) = \delta^{(3)}(\vec{x}). \tag{6}$$

1. Writing equation (6) in Fourier space, show that :

$$G(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + m^2}. \tag{7}$$

Is the Green's function unique ? Do we need to go through the contour-deforming procedure in this case ? Why is that the case ?

2. Perform the 3d momentum integral explicitly and show that :

$$G(\vec{x}) = \frac{e^{-m|\vec{x}|}}{4\pi|\vec{x}|} \tag{8}$$

*[Hint : choose spherical coordinate *wisely*, then perform the integral over angles, then use a contour integral for the final integration over $|\vec{k}|$.]*

4. The propagator as a Green function

1. Using the definition of the Feynman, retarded and advanced contours, show that :

$$G_F(x) = G_A(x) + iD_+(x), \qquad \textit{and} \qquad G_F(x) = G_R(x) + iD_-(x) \tag{9}$$

   where G_A and G_R are the advanced and retarded Green's functions, and $D_\pm$ are the positive and negarive frequency *Wightman functions*, defined by :

$$D_\pm(x) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{\mp i(\omega_p t - \vec{p}\cdot\vec{x})}}{2\omega_p}$$

   where $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$. Verify that $D_+(x) = D_-(-x)$.

2. Using the results above, show that

$$G_F(x - x') = i\Big(\theta(t - t')D_+(x - x') + \theta(t' - t)D_-(x - x')\Big). \tag{10}$$

3. By explicit calculation applying the differential KG operator in position space show that the right hand side of equation (10) is a Green's function for the Klein-Gordon equation.

4. In free Klein-Gordon theory, compute the two-point function :

$$\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle,$$

   and give its relation to $D_+(x - x')$ and $D_-(x - x')$. Use this to obtain the result :

$$G_F(x - x') = i\langle 0|T\Big(\hat{\phi}(x)\hat{\phi}(x')\Big)|0\rangle.$$

   where T denotes time-ordering.

   This gives the fundamental result that the Feynman Green's function coincides with the time-ordered two-point correlator.

5. Similarly, show that

$$G_R(x, x') = i\theta(t - t')\,\langle 0|\Big[\hat{\phi}(x), \hat{\phi}(x')\Big]|0\rangle. \tag{11}$$

5. Lorentz-invariant measure

Show that the integration measure

$$\frac{d^3p}{2\omega_p} \tag{12}$$

is Lorentz-invariant (where $\omega_p \equiv \sqrt{|\vec{p}|^2 + m^2}$).
[Hint : rewrite (12) as an integration measure over 4-momentum with suitable Dirac delta-functions]