CKM Matrix and CP violation

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CKM Matrix and CP Violation

1. Cabibbo-Kobayashi-Maskawa (CKM) Matrix
   1. Concept and definition
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   3. General parameterization
   4. Measurements, size and pattern of the CKM elements
   5. The Wolfenstein parameterization
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4. Experimental techniques for flavour tagging

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The CKM (Cabibbo, Kobayashi, Maskawa) Matrix

In the quark sector: weak Int. eigenstates ≠ flavour eigenstates

↔ States that participate in weak processes are linear combinations of flavour eigenstates
↔ Existence of 3X3 unitary matrix describing the mixing of quarks: the CKM Matrix

weak interaction eigenstates (≠ strong interaction eigenstates)
The CKM Formalism and CP Violation

Transition amplitude between, e.g., b and u quarks

\[ V_{ub} \]

Transition amplitude between, e.g., \( \bar{b} \) and \( \bar{u} \) anti-quarks

\[ V_{\bar{u}\bar{b}}^* \]

If the CKM matrix is not real (CKM phase) \( \Rightarrow V_{\bar{u}b}^* \neq V_{ub} \)

\( \Rightarrow \) different behavior of matter and anti-matter

\( \Rightarrow \) **CP violation!**

(questions: why are we talking about CP and not simply about C?)

Obviously, this single amplitude cannot give an observable CP violation. We must have a sum of amplitudes \( \Rightarrow \) contribution from a few processes

But is the CKM matrix complex?
The CKM matrix \( (V_{\text{CKM}}) \) is an \( n \times n \) complex matrix
\( \Rightarrow \) in general, contains \( 2n^2 \) real parameters.

\( V_{\text{CKM}} \) is unitary \( V^\dagger V \ (=VV^\dagger) = 1 \ \Rightarrow \ n^2 \) constraints \( \Rightarrow \ n^2 \) real parameters.

Each quark field has an arbitrary phase. As this phase cannot be observed and does not influence the system \( \Rightarrow \) physics is invariant under the transformation:

\[
V \rightarrow \begin{pmatrix}
e^{i\Phi_1^U} & 0 & \cdots & 0 \\
0 & e^{i\Phi_2^U} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & e^{i\Phi_n^U}
\end{pmatrix} V
\]

The overall phase cannot be fixed a-priori \( \Rightarrow \ 2n-1 \) phases can be removed from \( V_{\text{CKM}} \)
\( \Rightarrow \ n^2-(2n-1) = (n-1)^2 \) independent meaningful parameters

A practical way to construct a unitary matrix with the smallest number of phases:
- Start from an \( n \times n \) (real) rotation matrix \( \Rightarrow \frac{1}{2} n(n-1) \) rotation (mixing) angles
- Take the other parameters as phases (non-reducible \( \Leftrightarrow \) cannot be rotated away).
## Number of parameters in the CKM Matrix (II)

<table>
<thead>
<tr>
<th># generations</th>
<th># parameters</th>
<th># angles</th>
<th># non-reducible phases</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>(n-1)^2</td>
<td>n(n-1)/2</td>
<td>n(n+1)/2 – (2n-1) = (n-1)(n-2)/2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

No phase for two generations!

→ **At least 3 generations are needed to have the CKM phase and CP violation!**

The fact that there are 3 families (with neutrino mass < ½ M_Z) has been proven at LEP from the width of the Z mass peak.

A comment about the quark sector of the standard model:

10 free parameters in the flavour sector of the SM:

- 6 quark masses
- 4 CKM parameters
The 2008 Nobel Prize in Physics

was awarded to Kobayashi, Maskawa, and Nambu for their work on symmetry breaking and CP violation.

From the BaBar statement following the Nobel Prize:

[...] They found that it was very hard to construct a plausible explanation of CP violation in quark decays working with only these two generations of four quarks. Their brilliant insight of 1972 was to realize that by extending the number of generations to three — and hence the number of quarks from four to six — CP violation appears quite naturally. Thus their description of CP violation entailed the very bold prediction of two entirely new and unobserved types of quark, now called "top" (t) and "bottom" (b). Quite remarkably, these new quarks were indeed discovered experimentally, the b in 1977 and the t in 1995. More recently, Kobayashi and Maskawa's description of CP violation in quark decays was confirmed in detail by precision experiments at BaBar and Belle; their Nobel prize followed.
There is no unique parameterization of the CKM matrix. We can use, for example:

\[
V = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & \sin \theta_{23} \\
0 & -\sin \theta_{23} & \cos \theta_{23}
\end{bmatrix} \times \begin{bmatrix}
\cos \theta_{13} & 0 & \sin \theta_{13} e^{-i\delta} \\
0 & 1 & 0 \\
-\sin \theta_{13} e^{i\delta} & 0 & \cos \theta_{13}
\end{bmatrix} \times \begin{bmatrix}
\cos \theta_{12} & \sin \theta_{12} & 0 \\
-\sin \theta_{12} & \cos \theta_{12} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

This representation is the one used by the PDG (p. 211 in PDG 2016, removed from PDG 2018)

Remark: the number of possibilities is

\[(3!)_{\text{rotation permutations}} \times 3^\delta \times 2^{\delta=\pm1} = 36 \text{ possibilities}\]
Size of the elements and pattern of the Matrix

\[ |V_{\text{CKM}}| = \begin{pmatrix} 0.97434 & 0.22506 & 0.00357 \\ 0.22492 & 0.97351 & 0.0411 \\ 0.00875 & 0.0403 & 0.99915 \end{pmatrix} \pm \begin{pmatrix} 0.00011 & 0.00050 & 0.00015 \\ 0.00050 & 0.00013 & 0.0013 \\ 0.00032 & 0.0013 & 0.00005 \end{pmatrix} \]

(diagonal terms dominate: d~d’, s~s’ et b~b’)

We notice that, with \( \lambda = \sin \theta_c \approx 0.22 \)

\[ |V_{\text{CKM}}| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \]

The SM does not provide an explanation for this numerical pattern!
Wolfenstein Parametrization

Power series of $\lambda = \sin(\theta_{\text{cabibo}}) \approx 0.22$

At order $\lambda^3$:

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & \lambda \lambda^3 (\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 & \lambda \lambda^2 \\ A \lambda^3 (1 - \rho - i\eta) & -\lambda \lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

$\lambda = \sin \theta_c$
$A \sim 0.8$
$\rho \sim 0.20$
$\eta \sim 0.35$

At order $\lambda^5$:

$$V_{\text{CKM}} = V_{\text{CKM}}^{W3} + \begin{pmatrix} \frac{1}{2} A^2 \lambda^5 (1 - 2(\rho + i\eta)) & -\frac{1}{8} \lambda^4 (1 + 4A^2) & 0 \\ \frac{1}{2} A \lambda^5 (\rho + i\eta) & \frac{1}{2} A \lambda^4 (1 - 2(\rho + i\eta)) & -\frac{1}{2} A^2 \lambda^4 \\ \frac{1}{2} A^2 \lambda^5 (1 - 2(\rho + i\eta)) & -\frac{1}{8} \lambda^4 (1 + 4A^2) & 0 \end{pmatrix} + \mathcal{O}(\lambda^6)$$
Unitarity conditions

\[
VV^+ = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\times \begin{pmatrix}
V^*_{ud} & V^*_{cd} & V^*_{td} \\
V^*_{us} & V^*_{cs} & V^*_{ts} \\
V^*_{ub} & V^*_{cb} & V^*_{tb}
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}
\]

Diagonal relations (unitarity)

- \(V_{ud}V^*_{ud} + V_{us}V^*_{us} + V_{ub}V^*_{ub} = 1\)
- \(V_{cd}V^*_{cd} + V_{cs}V^*_{cs} + V_{cb}V^*_{cb} = 1\)
- \(V_{td}V^*_{td} + V_{ts}V^*_{ts} + V_{tb}V^*_{tb} = 1\)

Off-diagonal relations (orthogonality)

3 independent relations (3 are conjugates of 3 others)

- \(V_{ud}V^*_{cd} + V_{us}V^*_{cs} + V_{ub}V^*_{cb} = 0\)
- \(V_{ud}V^*_{td} + V_{us}V^*_{ts} + V_{ub}V^*_{tb} = 0\)
- \(V_{cd}V^*_{ud} + V_{cs}V^*_{us} + V_{cb}V^*_{ub} = 0\)
- \(V_{cd}V^*_{td} + V_{cs}V^*_{ts} + V_{cb}V^*_{tb} = 0\)
- \(V_{td}V^*_{cd} + V_{ts}V^*_{cs} + V_{tb}V^*_{cb} = 0\)
- \(V_{td}V^*_{ud} + V_{ts}V^*_{us} + V_{tb}V^*_{ub} = 0\)

from \(V^+V = 1\)

3 other independent relations

- \(V^*_{ud}V_{us} + V^*_{cd}V_{cs} + V^*_{td}V_{ts} = 0\)
- \(V^*_{ud}V_{ub} + V^*_{cd}V_{cb} + V^*_{td}V_{tb} = 0\)
- \(V^*_{us}V_{ub} + V^*_{cs}V_{cb} + V^*_{ts}V_{tb} = 0\)

The 6 orthogonality relations describe triangles in the complex plane
"The" Unitarity Triangle

The Unitarity Triangle

We notice that it's sides are comparable

It is usually divided by $V_{cd}V_{cb}^*$ (side of length 1)

Often called “the” unitarity triangle

CP Violation is possible in the Standard Model only if
$V_{\text{CKM}}$ is complex $\Leftrightarrow \overline{\eta} \neq 0 \Leftrightarrow$ Unitarity Triangle is not flat

We want to determine $\overline{\rho}$ and $\overline{\eta}$ experimentally by measuring the triangle sides and angles
Angles and apex of “The” Unitarity Triangle

\[ \bar{\rho} + i\bar{\eta} = -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \]

\[ \alpha = \text{Arg}\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}}\right) \]

\[ \beta = \text{Arg}\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}}\right) \]

\[ \gamma = \text{Arg}\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) \]

These are the exact coordinates of the apex. They slightly differ from the Wolfenstein parameterization, at \( O(\lambda^3) \)

By construction \( \alpha + \beta + \gamma = \pi \)

(only two independent angles)
Flavour oscillations in the neutral kaons system
(case with no CP violation)

- Amplitude of an instable particle (e.g. $K_S$):
  $$a_s(t) = a_s(0)e^{-\left(\frac{\Gamma_s}{\hbar}\right)t}$$

- Probability:
  $$\Gamma(t) = a_s(t)a_s^*(t) = a_s(0)a_s^*(0)e^{-\left(\frac{\Gamma_s}{\hbar}\right)t} = \Gamma(0)e^{-\left(\frac{\Gamma_s}{\hbar}\right)t}$$

- For the $K^0$-$\bar{K}^0$ system:
  $$\begin{align*}
  K_S : & \quad a_s(t) = a_s(0)e^{-\left(\frac{\Gamma_s}{\hbar}\right)t} \\
  K_L : & \quad a_L(t) = a_L(0)e^{-\left(\frac{\Gamma_L}{\hbar}\right)t}
  \end{align*}$$

$t = 0$: pure beam of $K^0$. Given that: $|K^0\rangle = \sqrt{\frac{1}{2}} (|K_S^0\rangle + |K_L^0\rangle) \Rightarrow a_L(0) = a_s(0) = \frac{1}{\sqrt{2}}$

At time $t$ (in natural units):
$$\Gamma\left(|K^0\rangle(t)\right) = \frac{(a_s(t) + a_L(t))}{\sqrt{2}} \cdot \frac{(a_s^*(t) + a_L^*(t))}{\sqrt{2}} = \frac{1}{4} \left\{ e^{-\Gamma_s t} + e^{-\Gamma_L t} + 2e^{-\frac{\Gamma_s + \Gamma_L}{2} t} \cos \Delta m t \right\}$$

- for $\bar{K}^0$ ($\Delta m = |m_L - m_S|$)

$$\Gamma\left(|K^0\rangle\right) - \Gamma\left(|\bar{K}\rangle\right) = e^{-\left[\left(\frac{\Gamma_s + \Gamma_L}{2}\right)t\right]} \cos \Delta m t$$

$$\Gamma\left(|K^0\rangle\right) + \Gamma\left(|\bar{K}\rangle\right) = \frac{1}{2} \left( e^{-\Gamma_s t} + e^{-\Gamma_L t} \right)$$

The $K^0$-$\bar{K}^0$ oscillation frequency is $\Delta m$
The experimental measurement for neutral kaons gives:

\[ \Delta m \equiv 3.52 \cdot 10^{-6} \text{ eV} \quad m_{K_L} > m_{K_s} \]

Very small mass difference (due to weak interaction)
We don’t have to worry about it…

Note that by measuring the frequency we can access experimentally a tiny mass difference \( \Delta m/m \sim 0.7 \cdot 10^{-14} \) !!!

Recall that this measurement gives access to some of the parameters of the SM: CKM matrix elements.

(it also provides information on the masses of the virtual quarks in the box)
Comparison of $K$, $B_d$ and $B_s$ Oscillations

- Oscillations (mixing) characterized by mass and lifetime differences between the two eigenstates of weak interaction.

- Differences between flavours:
  - $K$: very different states (because of the phase space difference)
  - $B_d$: Oscillation and decay are comparable
  - $B_s$: Rapid oscillations

Mind the scales!
And Finally D-Oscillations

Very slow oscillations
An experimental challenge!
Both BaBar and Belle observed mixing (Winter 2007)
Results are consistent with SM
Charm sector: only place where CP violation with down-type quarks in the mixing diagram can be explored.

- LHCb has now taken over these measurements
- CP violation in Charm decays was observed by LHCb in 2019

\[
x = \frac{m_1 - m_2}{\Gamma}\]
\[
y = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}\]
\[
\Gamma = \frac{1}{2}(\Gamma_1 + \Gamma_2)\]

SM: D mixing expected at \( \lesssim 1\% \) level

\[
x = (8.7 \pm 3.3) \times 10^{-3}\]
\[
y = (6.7 \pm 2.1) \times 10^{-3}\]

No-mixing point excluded at 5.7\( \sigma \)
## Time Evolution Plots (I)

### $K^0$

![Plot $K^0$ evolution](image1.png)

- $N(T) / N_0$ vs $T$
- $A_{\text{mix}}(T)$
- $T \equiv \Gamma_K t \approx t / 2 \tau_{K^0}$

### $B_d^0$

![Plot $B_d^0$ evolution](image2.png)

- $N(T) / N_0$ vs $T$
- $B_d^0 \to B_d^0$
- $T \equiv t / \tau_d$

### $B_s^0$

![Plot $B_s^0$ evolution](image3.png)

- $N(T) / N_0$ vs $T$
- $B_s^0 \to B_s^0$
- $B_s^0 \to B_s^0$
- $T \equiv t / \tau_s$

### Table of Decay Times and Mass Differences

<table>
<thead>
<tr>
<th>Particle</th>
<th>$\tau_L$</th>
<th>$\tau_H$</th>
<th>$\Delta m = m_H - m_L$ (hs$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^0$ (s$\bar{d}$)</td>
<td>$\sim 0.9 \times 10^{-10}$</td>
<td>$\sim 0.5 \times 10^{-7}$</td>
<td>$0.53 \times 10^{10}$ ($\sim$ a few $10^{-6}$ eV)</td>
</tr>
<tr>
<td>$D^0$ (c$\bar{u}$)</td>
<td>$\tau_H \sim \tau_L \sim 0.41 \times 10^{-12}$</td>
<td></td>
<td>$0.95 \times 10^{10}$</td>
</tr>
<tr>
<td>$B_d$ (b$\bar{d}$)</td>
<td>$\tau_H \sim \tau_L \sim 1.5$</td>
<td>$10^{-12}$</td>
<td>$0.51 \times 10^{12}$ ($3.4 \times 10^{-4}$ eV)</td>
</tr>
<tr>
<td>$B_s$ (b$\bar{s}$)</td>
<td>$\tau_H \sim \tau_L \sim 1.5$</td>
<td>$10^{-12}$</td>
<td>$17.76 \times 10^{12}$ ($1.12 \times 10^{-2}$ eV)</td>
</tr>
</tbody>
</table>

CDF, D0 2006
Figure 3.3: If one starts with a pure $P^0$-meson beam the probability to observe a $P^0$ or a $\bar{P}^0$-meson at time $t$ is shown, $\text{Prob}(t) = \frac{e^{-\Gamma t}}{2} \left( \cosh \frac{1}{2} \Delta \Gamma t \pm \cos \Delta m t \right)$. 
The $B^0$ mixing was observed for the first time in 1987 by the Argus collaboration:

$B^0 \rightarrow D^* \mu^+ \nu$  
$B^0 \rightarrow D^* \mu^+ \nu$

This predicted that $m(\text{top}) > 50 \text{ GeV}$!

---

B factories: (2005)

asymétrie: $\propto \cos(\Delta m_d t)$
Classification of CP Violation effects

Direct CP Violation (CP Violation in Decay):
$$\Gamma(X\rightarrow f) \neq \Gamma(\overline{X}\rightarrow \overline{f}) \quad (|\overline{A_f}| \neq |A_f|)$$

To measure it, only need to count events (e.g. for $B^0\rightarrow K^+\pi^-$)
Rates are different $\Leftrightarrow$ CP is violated
This is the only possible type CP violation in charged-particle and baryon decays

**CP violation in mixing:**
$$\Gamma(B^0\rightarrow \overline{B^0}) \neq \Gamma(\overline{B^0}\rightarrow B^0) \quad (|q/p|\neq 1)$$
N.B. unlike in neutral kaons, for $B^0$ and $B^0_s$ decays $|q/p|\approx 1$

**CP violation in the interference between decay and mixing:**
$$\Gamma(B^0\rightarrow f) \neq \Gamma(\overline{B^0}\rightarrow f) \quad (e.g. \text{ for } B^0\rightarrow J/\psi K_S)$$
may occur even if $|q/p|=1$ due to the phase of $q/p$

Analogy to “Double-Slit” experiment

*In the double-slit experiment*, there are two paths to the same point on the screen.
B tagging technique at B factories ($\Upsilon(4S)$)

$\Upsilon(4S)$ produces coherent $B\bar{B}$ pair:

$$\Delta t = \Delta z / \beta v c$$

$\Upsilon(4S)$

$e^{-} e^{+}$

$B_{CP}^{0}$

$B_{tag}^{0}$

$\beta v_{\Upsilon(4S)} = 0.56$

Proper time

Exclusive $B_{CP}$ meson and vertex reconstruction

$\Delta z \sim 260 \mu m$

$\psi \rightarrow \mu^{-} \mu^{+} \pi^{+} \pi^{-}$

$K_{S}$

B$_{tag}$ flavor and vertex reconstruction
B tagging technique at hadron colliders

Flavour Tagging: Determine B production flavours

SS Pion
SS Kaon
SS Proton
SS Pion BDT

Same Side

Opposite Side

PV

SS Kaon NNet
SS Proton NNet
SS Pion BDT

OS Kaon
OS K. NNet

OS Muon
OS Electron

OS Vertex Charge
OS Charm

B

$B^0$

$\pi^+$

$\bar{B}$

$\bar{b} \rightarrow c$

$\bar{b} \rightarrow X l^-$

$c \rightarrow s$

$J/\psi$

$K^*0$

$K^-$

$l^-$
Measurement of \( \sin(2\beta) \) with \( B^0 \rightarrow J/\psi K^0_S \)

- Final state accessible to \( B^0 \) and \( \bar{B}^0 \) → Time dependent asymmetry:

\[
A_{CP}(t) = \frac{\Gamma(\bar{B}^0(t) \rightarrow J/\psi K_S) - \Gamma(B^0(t) \rightarrow J/\psi K_S)}{\Gamma(B^0(t) \rightarrow J/\psi K_S) + \Gamma(B^0(t) \rightarrow J/\psi K_S)} = \sin(\Delta m_d t) - C \cos(\Delta m_d t)
\]

- \( C_f = 0 \)
- \( S_f = -\eta_{CP} \sin 2\beta \)

⇒ Extraction of \( \sin(2\beta) \) from \( A_{cp} \)
$\sin 2\beta$ measurement

[BABAR, PRD79, 072009 (2009)]

BaBar at ICHEP '08

$B \rightarrow (J/\psi, \psi(2S), \chi_{c1}, \eta_c) K_S^0$

$\sin 2\beta = 0.688 \pm 0.032$
Unitarity triangle measurements (2018)