Mid term exam

1 BaBar experiment

The BaBar experiment was an electron-positron collider aimed at studying CP violation. The collider (PEP-II) properties are given in Tab. 1.

Table 1: Properties of the PEP-II collider (and naming convention for the exercise

Property	value
Electron beam (oriented along $+\vec{z}$ axis and named A beam)	9 GeV
Positron beam (oriented along $-\vec{z}$ axis and named B beam)	3.1 GeV

We note throughout the exercise $\sqrt{s}$ the value of the total energy in the center of mass.
In this exercise, we consider the production of pair of particle-antiparticle ($F\bar{F}$):

$$e^+e^- \to F\bar{F}$$

This exercise is divided into 8 questions which are mostly independent from one another with the exception of question 5.

1. *Cross section master formula*

$$d\sigma = \frac{1}{2\,E_A\;2\,E_B|\beta_A - \beta_B|} \times |\mathcal{M}|^2 \times (2\pi)^4\delta^{(4)}(p_A + p_B - \sum_{i=1}^{i\leq f} k_i)\prod_{i=1}^{i\leq f}\frac{d^3\vec{k}_i}{(2\pi)^3\;2\,E_i}, \tag{1}$$

- Flux term (not Lorentz invariant): $\frac{1}{2\,E_A\;2\,E_B|\beta_A-\beta_B|}$.
- Matrix element (Lorentz invariant): $|\mathcal{M}|^2$
- Phase space (Lorentz invariant): $(2\pi)^4\delta^{(4)}(p_A + p_B - \sum_{i=1}^{i\leq f} k_i) \times \prod_{i=1}^{i\leq f}\frac{d^3\vec{k}_i}{(2\pi)^3\;2\,E_i}$

2. *Phase space term dLIPS*

   (a) When we place ourselves in the CM, integrating over $d^3\vec{k}_2$ fixes $|\vec{k}_2| = |\vec{k}_1|$ via the δ-function $(2\pi)^3\delta^{(3)}(\vec{0} - \vec{k}_1 - \vec{k}_2)$, it also includes a factor of (4π) due to the integration over $d\Omega_2$.

$$\int_{\vec{k}_2} dLIPS = (2\pi)\delta(\sqrt{s} - E_1^* - E_2^*)\frac{d^3\vec{k}_1}{(2\pi)^3\;2\,E_1^*} \times \frac{4\pi}{2\,E_2^*},$$

   where $*$ refers to CM quantities.

   (b) We note k_* the fixed value of $|\vec{k}_1|$ in the center of mass. The remaining δ-function fixes k_* via the conservation of energy. Because the 2 final particles have the same mass m_F and the same absolute 3-momentum, they get the same energy in the center of mass $E_1^* = E_2^* = \sqrt{m_F^2 + k_*^2}$.

$$\begin{aligned} E_1^* + E_2^* &= 2\;E_1^* = \sqrt{s} \\ &= 2\;\sqrt{m_F^2 + k_*^2} \end{aligned} \tag{2}$$

   which gives immediately:

$$k_* = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4\,m_F^2}{s}} \tag{3}$$

(c) The remaining $dLIPS$ can therefore be written as

$$
\begin{aligned}
dLIPS &= (2\pi)\delta(2E_1^* - \sqrt{s})\frac{k^2\ dkd\Omega}{(2\pi)^3 2\,E_1^*}\frac{1}{2\,E_1^*} \\
&= \frac{\delta(k-k_*)}{2\frac{k}{E_1^*}}\frac{k^2\ dk\ d\Omega_*}{(2\pi)^2 4\,(E_1^*)^2} \\
&= \frac{1}{32\pi^2}\ \frac{k}{E_1^*}\delta(k-k^*)dk\ d\Omega \\
&= \frac{1}{32\pi^2}\ \frac{2\,k}{\sqrt{s}}\delta(k-k^*)dk\ d\Omega
\end{aligned} \tag{4}
$$

Integrating over dk and using $k_* = \frac{\sqrt{s}}{2}\sqrt{1-\frac{4\,m_F^2}{s}}$

$$
\begin{aligned}
dLIPS &= \frac{1}{32\pi^2}\ \frac{2\,k_*}{\sqrt{s}}d\Omega \\
dLIPS &= \frac{1}{32\pi^2}\ \sqrt{1-\frac{4\,m_F^2}{s}}d\Omega
\end{aligned} \tag{5}
$$

3. *Matrix element $\mathcal{M}$,* part completely independent.

   (a) see course

   (b)

$$
i\mathcal{M} = i^2(4\pi)\alpha^2\bar{u}(k_1)\gamma^\nu v(k_2)\frac{\eta_{\mu\nu}}{(k_1+k_2)^2+i\epsilon}\bar{v}(p_B)\gamma^\mu u(p_A)
$$

4. *Flux term $\mathcal{F}$*

   (a) The Lorentz transformation to the CM

$$
\Lambda_\mu^\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}
$$

   where γ and β are the CM parameters. Therefore:

$$
E_A* = \gamma(E_A - \beta(p_A)_z \approx \gamma(E_A - \beta E_A) = E_A\gamma(1-\beta)
$$

   Similarly , but $(p_B)_z$ is oriented along $-\vec{z}$:

$$
E_B* = \gamma(E_A + \beta E_A) = E_A\gamma(1+\beta)
$$

   (b)

$$
\begin{aligned}
E_A^*E_B^* &= \gamma^2 E_A E_B(1-\beta)(1+\beta) \\
E_A^*E_B^* &= E_A E_B(1-\beta^2)\gamma^2 \\
E_A^*E_B^* &= E_A E_B
\end{aligned} \tag{6}
$$

   (c) Using this property and the fact tha in the CM $E_A^* = E_B^* = \sqrt{s}/2$

$$
\mathcal{F} = \frac{1}{2E_A 2E_B|\beta_A - \beta_B|} = \frac{1}{8E_A^*\ E_B^*} = \frac{1}{2s}
$$

5. *Differential cross section,*

   (a) Putting all the pieces together we find that:

$$
\begin{aligned}
d\sigma &= \frac{1}{2s}|\mathcal{M}|^2\frac{1}{32\pi^2}\;\frac{2\,k_*}{\sqrt{s}}d\Omega \\
d\sigma &= \frac{1}{64\pi^2 s}|\mathcal{M}|^2\frac{2\,k_*}{\sqrt{s}}d\Omega \\
d\sigma &= \frac{1}{64\pi^2 s}|\mathcal{M}|^2\;\sqrt{1-\frac{4\,m_F^2}{s}}d\Omega
\end{aligned}
\tag{7}
$$

   (b) Using the matrix element formula:

$$
\begin{aligned}
\int d\Omega|\mathcal{M}|^2 &= \int d\cos\theta d\phi|\mathcal{M}|^2 \\
&= (2\pi)\int d\cos\theta\;|\mathcal{M}|^2 \\
&= (2\pi)(4\pi)^2\alpha^2\left\{\left(1+\frac{4\;m_F^2}{s}\right)\times 2+\left(1-\frac{4\;m_F^2}{s}\right)\times\frac{2}{3}\right\} \\
&= (4\pi)^3\alpha^2\left(\frac{4}{3}+\frac{4\;m_F^2}{s}\frac{2}{3}\right) \\
&= 64\pi^3\alpha^2\frac{4}{3}\left(1+\frac{1}{2}\frac{4\;m_F^2}{s}\right)
\end{aligned}
\tag{8}
$$

   which gives:

$$
\begin{aligned}
\sigma_{tot} &= \frac{1}{64\;\pi^2 s}\sqrt{1-\frac{4\,m_F^2}{s}}\;\;64\pi^3\alpha^2\frac{4}{3}\left(1+\frac{1}{2}\frac{4\;m_F^2}{s}\right) \\
\sigma_{tot} &= \frac{4\pi\;\alpha^2}{3\;s}\sqrt{1-\frac{4\,m_F^2}{s}}\left(1+\frac{1}{2}\frac{4\;m_F^2}{s}\right)
\end{aligned}
\tag{9}
$$

6. *Kinematics of the production.* This part is completely independent.

   (a) Using Tab. 1, compute the value of $\sqrt{s}$.

$$
\begin{aligned}
\sqrt{s} &= \sqrt{(p_A+p_B)^2} = \sqrt{(E_A+E_B)^2-(p_A+p_B)_z^2} \\
\sqrt{s} &= \sqrt{(9+3.1)^2-(9-3.1)^2}\text{ GeV} \\
\sqrt{s} &= 10.564\text{ GeV}
\end{aligned}
\tag{10}
$$

   (b) GIven the width of the $\Upsilon(4S)$ resonance, the reaction e^+e^- at $\sqrt{s}=10.56$ GeV can occur via the resonance. This means that the total cross section is enhanced via the resonance compare to expectation from LO (reminder a resonance is actually due to a pole in the propagator due to NLO effects).

   (c) The $\Upsilon(4S)$ decays mostly to B^+B^- and $B^0\bar{B}^0$ (about 50 % branching ratio for each channel).

7. *The different $F\bar{F}$ productions.*

   (a) In In the approximation $2\,m_F \ll \sqrt{s}$, Eq. 3 gives:

$$
\sigma_{tot} \approx \frac{4\pi}{3}\frac{\alpha^2}{s}
$$

This approximation is justified for muons $m_\mu = 0.1$ GeV $\ll 10.56$ GeV. s being in GeV we need to convert it a to an area using $\hbar c$. This gives:

$$
\begin{aligned}
\sigma(e^+e^- \to \mu^+\mu^- &\approx \frac{4\pi}{3}\frac{(1/137)^2}{10.56^2}(\hbar c)^2 \\
\sigma(e^+e^- \to \mu^+\mu^- &\approx \frac{4\pi}{3}\frac{(1/137)^2}{10.56^2}\ (200\ 10^{-3}\ 10^{-15})^2\ m^2 \\
\sigma(e^+e^- \to \mu^+\mu^- &\approx \frac{4\pi}{3}\frac{(1/137)^2}{10.56^2}\ (200\ 10^{-3}\ 10^{-15})^2 10^{28}\ \mathrm{b} \\
\sigma(e^+e^- \to \mu^+\mu^- &\approx \frac{4\pi}{3}\frac{(1/137)^2}{10.56^2}\ 4\ 10^{-32}\ 10^{28}\ \mathrm{b} \\
\sigma(e^+e^- \to \mu^+\mu^- &\approx 8\ 10^{-6} 10^{-32} 10^{28}\ \mathrm{b} \\
\sigma(e^+e^- \to \mu^+\mu^- &\approx 8\ 10^{-10}\ \mathrm{b} \\
\sigma(e^+e^- \to \mu^+\mu^- &\approx 0.8\ \mathrm{nb}
\end{aligned} \tag{11}
$$

Therefore the order of magnitude from LO only is correct. The different with the total cross section (1.15 nb) is due to NLO QED correction (especially with the emission of additional photon in the intial/final state).

(b) Neglecting QCD effect we can predict (the quark charge intervene only in the final vertex):

$$
\begin{aligned}
\sigma(u\bar{u}) &= Q_u^2 N_c \sigma(\mu^+\mu^-) = 4/3\sigma(\mu^+\mu^-) = 1.53\ \mathrm{nb} \\
\sigma(d\bar{d}) &= Q_d^2 N_c \sigma(\mu^+\mu^-) = 1/3\sigma(\mu^+\mu^-) = 0.38\ \mathrm{nb}
\end{aligned} \tag{12}
$$

where Q_q is the charge of quark q and $N_c = 3$ is the number of colors.
This predictions are decently matching the cross section from the table. The differences coming essentially from QCD NLO effect.

(c) At LO QED we expect

$$
\frac{\sigma(e^+e^- \to b\bar{b})}{\sigma(e^+e^- \to d\bar{d})} = \sqrt{1 - \frac{4\,m_F^2}{s}}\left(1 + \frac{1}{2}\frac{4\ m_F^2}{s}\right)
$$

where the last part is due to m_b mass effect in the phase space and matrix element. Using $m_b = 4.5$ GeV, we expect:

- b mass in phase space only: $\sqrt{1 - \frac{4\,m_b^2}{s}} = 0.52$
- b mass in matrix element only: $\left(1 + \frac{1}{2}\frac{4\ m_F^2}{s}\right) = 1.36$
- we therefore would expect

$$
\frac{\sigma(e^+e^- \to b\bar{b})}{\sigma(e^+e^- \to d\bar{d})} = 0.71
$$

(d) The actual ratio is $\frac{\sigma(e^+e^- \to b\bar{b})}{\sigma(e^+e^- \to d\bar{d})} = 2.775 \gg 0.71$. The difference is due to the production of the $\Upsilon(4S)$ resonance which is a NLO effect in the photon propagator. Since the $\Upsilon(4S)$ decays exclusively to $b\bar{b}$ pairs (see question II.6.c) this enhances only the $b\bar{b}$ production.

8. *Available statistics.* This part is completely independent.

(a) The instantaneous luminosity of the machine was $\mathcal{L} \approx 10 \times 10^{33}\mathrm{cm}^{-2}\mathrm{s}^{-1}$. So for a month:

$$
\begin{aligned}
L &= \mathcal{L} \times 20 \times 24 \times 3600 \\
L &= 4.8 \times 3.6 \times 10^{34} 10^{5}\mathrm{cm}^{-2} \\
L &= 17.28\ 10^{39} 10^{-24}\ \mathrm{b}^{-1} \\
L &= 17.28\ 10^{39} 10^{-24} 10^{-15}\ \mathrm{fb}^{-1} \\
L &\approx 17.3\ \mathrm{fb}^{-1}
\end{aligned} \tag{13}
$$

(b)

$$N_{b\bar{b}} = \sigma(b\bar{b}) \times L_{tot} = 1.11\ 10^6\text{fb} \times 433\text{fb}^{-1} = 480\ 10^6 \tag{14}$$

Plugging the branching fractions:

$$\begin{aligned} N_{B^+B^-} &== \sigma(b\bar{b}) \times L_{tot}\ \mathcal{B}(\Upsilon(4S) \to B^+B^-) = 480 \times 0.514 \times 10^6 = 246.7\ 10^6 \\ N_{B^0\overline{B}^0} &== \sigma(b\bar{b}) \times L_{tot}\ \mathcal{B}(\Upsilon(4S) \to B^0\overline{B}^0) = 480 \times 0.486 \times 10^6 = 233.3\ 10^6 \end{aligned} \tag{15}$$