1 The proton structure

In this part, we will focus on studying the proton structure using an electron probe. We will seek to define and determine the proton radius (the typical size of a proton is $R_p \approx 1$ fm.

(1) What is the typical electron energy you would need to probe the proton structure? What is the β,γ for such an electron, are we in the relativistic regime?

(2) From $\partial_\mu F^{\mu\nu} = j^\nu$, where $j^\nu \equiv (Q\rho(x), Q\vec{j}(x))$ is the charge current density, and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, find back the Poisson equation (assuming the charge distribution $\rho(x)$ is time-independent, $\rho(x) \equiv \rho(\vec{x})$:

$$\nabla^2 V(x) = -Q\ \rho(x) = -Q\ \rho(\vec{x})$$

with $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$ and Q the charge of the proton ($Q = -e$), note that $\rho(x)$ is a charge density distribution so $\iiint d^3 x \rho(\vec{x}) = 1$.

Writing the Fourier transform of $V(x)$, $\tilde{V}(q)$, and the one of $\rho(x)$, $\tilde{\rho}(q)$, use the Poisson equation to show that:

$$\tilde{V}(q) = \frac{Q}{|\vec{q}|^2}\ \tilde{\rho}(q)$$

If we consider a punctual charge *i.e.*$\rho(q) = \delta^3(\vec{x})$, find that the Fourier transform of a Coulomb field if $\tilde{V}_{\text{Clb}}$:

$$\tilde{V}_{\text{Clb}}(q) = \frac{Q}{|\vec{q}|^2}$$

(3) We consider the *time-independent* A^μ potential created by the proton as a classical field. Draw the Feynamn diagram associated to this interaction in the semi-classical approach. Write the corresponding matrix element (note: remember that in this case $\tilde{A}(q) = \delta(q^0)\tilde{A}(\vec{q})$ and we do not consider $\delta(q^0)$ in the matrix element).

Use the notations:

- incoming electron 4-momentum $p_1 \equiv (E, 0, 0, p)$,
- outgoing electron 4-momentum $k_1 \equiv (E', p' \sin\theta, 0, p' \cos\theta)$.
- ξ_s the spinor basis for the incoming electron, so the corresponding Dirac bi-spinor $u_s(p)$,

- $\eta_{s'}$ the spinor basis for the outgoing electron, so the corresponding Dirac bi-spinor $u_{s'}(p')$.
- q the transferred momentum to the photon: $q^\mu = (k_1 - p_1)^\mu$.

Simplify the formula Eq. **??** for the cross-section in this case by integrating over d^3p_f with the dirac $\delta(E_f - E_i)$. What does it imply in the matrix element for E' and $p' \equiv |\vec{p}'|$?

(4) Write q^2 as a function of the variables (p, θ).

(5) Since we want to use relativistic electrons, we can not use the approximation used in the Rutherford scattering formula. Let's consider the proton as a point-like particle ($\Rightarrow$ Coulomb potential). Use the helicity basis of the outgoing and incoming electrons from Eq. 2.4. Forcing $E' = E$ and $p' = p$ compute the different matrix elements (without forcing $\beta = 1$) as a function of $\eta_+, \eta_-, \xi_+, \xi_-$:

- $\mathcal{M}(+,+)$ incoming helicity$+$, outgoing helicity$+$,
- $\mathcal{M}(+,-)$ incoming helicity$+$, outgoing helicity$-$,
- $\mathcal{M}(-,+)$ incoming helicity$-$, outgoing helicity$+$,
- $\mathcal{M}(-,-)$ incoming helicity$-$, outgoing helicity$-$.

(6) In the limit $\beta = 1$ you should find that $\mathcal{M}(+,-) = \mathcal{M}(-,+) = 0$. Can you justify *without calculations* using the fact that in limit $\beta = 1$, $u_- \to u_L$ and $u_+ \to u_R$?

(7) We now need to use to compute the different matrix elements using the actual spinors ξ_+, ξ_- and η_+, η_-.

- (7a) Express the helicity operator $h2(p)$ and justify that $\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
- (7b) Express the helicity operator $h2(p')$ as a function θ, demonstrate that $\eta_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$, guess or find the value of η_-.

(8) Using these expressions of the spinor basis, one compute the final cross section by summing over the final spin and averaging over the initial spin so:

$$d\sigma_{\text{tot}} = \frac{1}{2}\left[d\sigma(+,+) + d\sigma(+,-) + d\sigma(-,+) + d\sigma(+,+)\right]$$

where $\sigma(a,b)$ corresponds to the matrix element from (5). Show that the final cross-section is (using $Q = -Ze$ for the proton $Z = 1$):

$$\frac{d\sigma_{\text{tot}}}{d\Omega} = \frac{Z^2\,\alpha^2}{4\,p^2\,\beta^2\,\sin^4\frac{\theta}{2}}\left(1 - \beta^2\,\sin^2\frac{\theta}{2}\right) \tag{1.1}$$

This is named the Mott cross section.

(9) Consider that the proton has an extended spatial charge $\rho(x)$. Show that the scattering amplitude is modified as follow:

$$\frac{d\sigma_{\text{tot}}}{d\Omega} = \left[\frac{d\sigma_{\text{tot}}}{d\Omega}\right]_{\text{Mott}} \times |G_e(q^2)|^2 \tag{1.2}$$

Express $G_e(q^2)$ as a function of $r\tilde{h}o(q)$?

(10) In this case, one can defined the charge radius of the proton from the spatial distribution as the RMS (Root mean squared) of the ρ distribution, *i.e.*:

$$R = \sqrt{<|\vec{x}|^2>} \qquad \text{with} <|\vec{x}|^2> = \iiint |\vec{x}|^2 \rho(\vec{x}) d^3\, x$$

(10a) If the charge distribution is exponential, *i.e.*:

$$\rho(\vec{x}) = N \times \; e^{-\frac{r}{r_0}}$$

where N is a normalisation factor (such that $\iiint d^3\, x \rho(\vec{x}) = 1$). Integrating by part (note that you do not need to compute N before hand) , show that:

$$R = \sqrt{12}\, r_0$$

(10b) Show that the Fourier transform of $N \times \; e^{-\frac{r}{r_0}}$ is :

$$\tilde{\rho}(\vec{q}) \propto \left(1 - r_0^2\, q^2\right)$$

with $q^2 = -|\vec{q}|^2$.

(11) From these results and equation 1.2, describe an experimental procedure to measure $G_e(q^2)$ and thus the size of the proton. You have at your disposal a small size detector able to detect a e^- particle (but not to measure their energy), an electron beam and a target of hydrogen.

(12) The $G_E(q^2)$ curve extracted from elastic scattering data as well as a fit to this data is given on Fig. 1.1. What do you conclude on the charge distribution in a proton and its average charge radius in fm ? NB: the unit in the plot is $0.71\ \text{GeV}^{-2}$

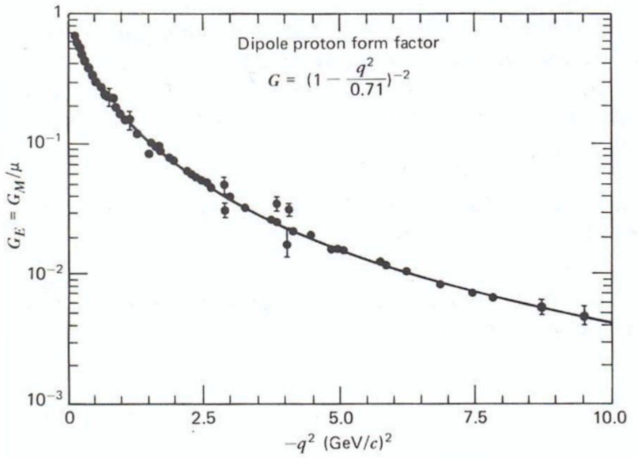

Fig. 1.1: Proton electric form factor $G_E(q^2)$ extracted from experimental elastic scattering data.

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γ and Pauli matrices :

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \vec{\sigma} \equiv \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \tag{2.1}$$

The bi-spinor conjugates are $\bar{u} \equiv u^\dagger \gamma^0$ and $\bar{v} = v^\dagger \gamma^0$. The relation between the Fourier transform $\tilde{f}(q)$ of a function $f(x)$ and the function itself are give by:

$$\begin{aligned} \mathcal{F}(f(x)) &\equiv \tilde{f}(q) \\ f(x) &= \int \frac{d^4\, q}{(2\pi)^4} e^{-i\, q.x} \tilde{f}(q) \\ \tilde{f}(q) &= \int d^4\, x\; e^{i\, q.x} f(x) \end{aligned} \tag{2.2}$$

A useful basis to express the Dirac bi-spinor is the helicity basis where $\eta_s \equiv (\eta_+, \eta_-)$ $\eta_{+/-}$ corresponding to the helicity $+1/2$ or $-1/2$. By definition, $\eta_{+/-}$ are eigenvectors of the helicity operator

$$\begin{aligned} h_2(p) &\equiv \frac{\vec{p}.\vec{\sigma}}{|\vec{p}|} \\ h_2(p)\ \eta_\pm &= +\eta_\pm \end{aligned} \tag{2.3}$$

In this basis, one can write the Dirac bi-spinor:

$$u_+(p) = \sqrt{\gamma\, m} \begin{pmatrix} \sqrt{1-\beta}\ \eta_+ \\ \sqrt{1+\beta}\ \eta_+ \end{pmatrix} \quad ; \qquad u_-(p) = \sqrt{\gamma\, m} \begin{pmatrix} \sqrt{1+\beta}\ \eta_- \\ \sqrt{1-\beta}\ \eta_- \end{pmatrix} \tag{2.4}$$

where γ and β are the corresponding β, γ of the particle, *i.e.*$p \equiv (E, \vec{p})$ such that $E = \gamma m$ and $|\vec{p}| = \gamma \beta m$. We remind the time-independent phase space in the semi-classical approach:

$$d\sigma = \frac{1}{2\, E_i\, \beta_i} \left| \mathcal{M}(i \to f) \right|^2 (2\pi) \delta(E_f - E_i)\, \frac{d^3 p_f}{(2\pi)^3\ 2\ E_f} \tag{2.5}$$

3 Corrections

(1) The size of a proton is 1fm, the DeBroglie wave length associated to the electron is such that $E_e = \frac{2\pi\hbar c}{\lambda_e} = 1/(1\text{fm})$, using $\hbar.c = 200$ MeV.fm, the corresponding energy is therefore $E_e \approx 1$ GeV (accept answer starting from 200 MeV).

$$\begin{aligned} E_e &= \gamma m_e & \Rightarrow \quad & \gamma = E_e/m_e \approx 1800 \\ 1/\gamma^2 &= 1-\beta^2 & \Rightarrow \quad & \beta = \sqrt{1-1/\gamma^2} \approx 1 \;-\; 0.15.10^{06} \end{aligned} \tag{3.1}$$

we are indeed in an ultra-relativistic regime.

(2) The maxwell equations are given for the zero component (we use time independence $\partial_t A^i = 0$).

$$\begin{aligned} \rho(x) = j^0 &= \partial_\mu F^{\mu 0} \\ &= \partial_i F^{i\,0} = \partial_i(\partial^i A^0 - \partial^0 A^i) \\ &= \partial_i \partial^i A^0 = -(\partial_i)^2 A^0 \\ &= -\nabla^2 V(x) \end{aligned} \tag{3.2}$$

we have by definition

$$V(x) = \iiint \frac{d^4q}{(2\pi)^4}\left[e^{-i\,q.x}\tilde{V}(q)\right]$$

Thus

$$\begin{aligned} \nabla^2 V(x) &= \int \frac{d^4q}{(2\pi)^4}\left[\left((\partial_i)^2\, e^{-i\,q.x}\right)\tilde{V}(q)\right] \\ \nabla^2 V(x) &= \int \frac{d^4q}{(2\pi)^4}\left[e^{-i\,q.x}(-i\,q_i)^2\tilde{V}(q)\right] \\ \nabla^2 V(x) &= -Q\rho(x) = \int \frac{d^4q}{(2\pi)^4}\left[e^{-i\,q.x} r\tilde{h}o(q)\right] \\ \Rightarrow \tilde{V}(q) &= -Q\frac{\tilde{\rho}(q)}{i^2(q_i)^2} = Q\,\frac{\tilde{\rho}(q)}{(-q^i)^2} \\ \Rightarrow \tilde{V}(q) &= \frac{Q}{|\vec{q}\,|^2}\,\tilde{\rho}(q) \end{aligned} \tag{3.3}$$

For the Coulomb field we have a Dirac distribution of charges, therefore $\tilde{\rho}_{\text{Clb}}(q) = 1$.

(3) Identical to the Rutherford scattering exercise.

The matrix element can be written:

$$i\mathcal{M}(s,s') = \bar{u}_{s'}(p')\gamma^0 u_s(p)\tilde{A}_0(q)(-i\,e)$$

Because the in and out particle have the same mass $\delta(E_f - E_i)$ will impose $p_f = p_i$ (just due to $E_i^2 = E_f^2$):

$$\begin{aligned}
d\sigma &= \frac{1}{2\,E_i\,\beta_i}\,|\mathcal{M}(i\to f)|^2\,(2\pi)\delta(E_f - E_i)\,\frac{d^3p_f}{(2\pi)^3\,2\,E_f}\\
&= \frac{1}{2\,E_i\,\beta_i}\,|\mathcal{M}(i\to f)|^2\,(2\pi)\delta(E_f(p_f) - E_i)\,\frac{p_f\,dp_f d\Omega}{(2\pi)^3\,2\,E_f}\\
&= \frac{1}{2\,E_i\,\beta_i}\,|\mathcal{M}(i\to f)|^2\,(2\pi)\frac{\delta(p_f - p_i)}{p_f/E_f}\,\frac{p_f^2\,dp_f d\Omega}{(2\pi)^3\,2\,E_f}\\
&= \frac{p_i}{2\,E_i\,\beta_i}\,|\mathcal{M}(p_f = p_i)|^2\frac{d\Omega}{(2\pi)^2\,2}\\
&= \frac{1}{(4\pi)^2}|\mathcal{M}(p_f = p_i)|^2 d\Omega
\end{aligned} \tag{3.4}$$

(4) Transferred 4-momentum:

$$\begin{aligned}
q^2 &= (k_1 - p_1)^2 = (E' - E)^2 - |\vec{p}\,' - \vec{p}\,|^2\\
&= 0 - 2\,|\vec{p}\,|^2(1 - \cos\theta) \qquad \text{using } E' = E \text{ and } p' = p \text{ forced from } \delta E_f - E_i\\
&= -\,p^2\,4\,\sin^2\frac{\theta}{2}
\end{aligned} \tag{3.5}$$

(5) We use the fact $\beta' = \beta$ and $\gamma' = \gamma$ from $\delta(E_f - E_i)$.

$$\begin{aligned}
\mathcal{M}(+,+) &= -ie\tilde{A}^0(q)\;\bar{u}_+(p')\gamma^0\,u_+(p)\\
&= -ie\tilde{A}^0(q)\;u_+^\dagger(p')(\gamma^0)^2\,u(p) = -ie\tilde{A}^0(q)\;u_+^\dagger(p')\,u_+(p)\\
&= -ie\tilde{A}^0(q)\;\gamma\,m\;\begin{pmatrix}\sqrt{1-\beta}\;\eta_+^\dagger & \sqrt{1+\beta}\;\eta_+^\dagger\end{pmatrix}\begin{pmatrix}\sqrt{1-\beta}\;\xi_+\\ \sqrt{1+\beta}\;\xi_+\end{pmatrix}\\
&= -ie\tilde{A}^0(q)\;(2\,\gamma\,m)\;\eta_+^\dagger.\xi_+
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\mathcal{M}(+,-) &= -ie\tilde{A}^0(q)\;\bar{u}_-(p')\gamma^0\,u_+(p)\\
&= -ie\tilde{A}^0(q)\;u_-^\dagger(p')(\gamma^0)^2\,u_+(p) = -ie\tilde{A}^0(q)\;u_-^\dagger(p')\,u_+(p)\\
&= -ie\tilde{A}^0(q)\;\gamma\,m\;\begin{pmatrix}\sqrt{1+\beta}\;\eta_-^\dagger & \sqrt{1-\beta}\;\eta_-^\dagger\end{pmatrix}\begin{pmatrix}\sqrt{1-\beta}\;\xi_+\\ \sqrt{1+\beta}\;\xi_+\end{pmatrix}\\
&= -ie\tilde{A}^0(q)\;(2\,\gamma\,m)\;\sqrt{1-\beta^2}\;\eta_-^\dagger.\xi_+
\end{aligned} \tag{3.7}$$

And eventually

$$
\begin{aligned}
\mathcal{M}(+,+) &= -ie\tilde{A}^0(q)\ (2\,\gamma\,m)\ \eta_+^\dagger.\xi_+ \\
\mathcal{M}(+,-) &= -ie\tilde{A}^0(q)\ (2\,\gamma\,m)\ (1-\beta^2)\ \eta_-^\dagger.\xi_+ \\
\mathcal{M}(-,+) &= -ie\tilde{A}^0(q)\ (2\,\gamma\,m)\ (1-\beta^2)\ \eta_+^\dagger.\xi_- \\
\mathcal{M}(-,-) &= -ie\tilde{A}^0(q)\ (2\,\gamma\,m)\ \eta_-^\dagger.\xi_-
\end{aligned}
\tag{3.8}
$$

(7a) $h2(p) = \sigma^3$ hence the value of ξ_+ and ξ_-

$$
h2(p) = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(7b) $h2(p') = \vec{p}'\vec{\sigma}/p' = \sin\theta\,\sigma^1\ +\ \cos\theta\,\sigma^3$:

$$
h2(p') = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}
$$

Then we immediately find that:

$$
h2(p')\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ \sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2} \end{pmatrix} = + \begin{pmatrix} \cos(\theta-\theta/2) \\ \cos(\theta-\theta/2) \end{pmatrix}
$$

So $\eta_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$. The fact that $\eta_+^\dagger\eta_- = 0$ gives straighhtforwardly:

$$
\eta_- = \begin{pmatrix} -\sin\frac{\theta}{2} \\ +\cos\frac{\theta}{2} \end{pmatrix}
$$

Note: the sign is $\sin$ and $\cos$ term can be inverted without any impact.

(8) Using the previous spinors we get directly

$$
\begin{aligned}
|\mathcal{M}(+,+)|^2 &= e^2|\tilde{A}^0(q)|^2\ (2\,\gamma\,m)\ \cos^2\frac{\theta}{2} \\
|\mathcal{M}(+,-)|^2 &= e^2|\tilde{A}^0(q)|^2\ (2\,\gamma\,m)\ (1-\beta^2)\ \sin^2\frac{\theta}{2} \\
|\mathcal{M}(-,+)|^2 &= e^2|\tilde{A}^0(q)|^2\ (2\,\gamma\,m)\ (1-\beta^2)\ \sin^2\frac{\theta}{2} \\
|\mathcal{M}(-,-)|^2 &= e^2|\tilde{A}^0(q)|^2\ (2\,\gamma\,m)\ \cos^2\frac{\theta}{2}
\end{aligned}
\tag{3.9}
$$

And therefore

$$
\begin{aligned}
\frac{d\sigma_{\text{tot}}}{d\Omega} &= \frac{1}{(4\pi)^2}\, e^2\, |\tilde{A}^0(q)|^2\, (4\,\gamma^2\, m^2)\, \left(\cos^2\frac{\theta}{2} + (1-\beta^2)\ \sin^2\frac{\theta}{2}\right) \\
&= \frac{1}{(4\pi)^2}\, e^2\, \frac{Q^2}{p^4\, 16\, \sin^4\frac{\theta}{2}} (4\,\gamma^2\, m^2)\, \left(1-\beta^2\ \sin^2\frac{\theta}{2}\right) \\
&= \frac{Z^2\,\alpha^2}{4\, p^2\, \beta^2\, \sin^4\frac{\theta}{2}}\, \left(1-\beta^2\ \sin^2\frac{\theta}{2}\right)
\end{aligned}
\tag{3.10}
$$

(9) From (2) $\tilde{V}(q) = \tilde{V}_{\text{Clb}}(q)\ \tilde{\rho}(q)$ which implies directly

$$
\frac{d\sigma_{\text{tot}}}{d\Omega} = \left[\frac{d\sigma_{\text{tot}}}{d\Omega}\right]_{\text{Mott}} \times |\tilde{\rho}(\vec{q})|^2
$$

(10a) N is such that

$$
\iiint \rho(\vec{x}) d^3\, x = N \times \left(\int_{r=0}^{\infty} r^2 e^{-\frac{r}{r_0}}\, dr\right) \left(\iint d(\cos\theta) d\phi\right) = 1
$$

Thus, integrating $< |\vec{x}|^2 >$ by part 2 times:

$$
\begin{aligned}
< |\vec{x}|^2 > &= \iiint |\vec{x}|^2 \rho(\vec{x}) d^3\, x = \iiint r^2 \rho(r) r^2 dr d(\cos\theta) d\phi \\
&= N \left\{\iint d(\cos\theta) d\phi\right\} \int_{r=0}^{\infty} r^4 e^{-\frac{r}{r_0}}\, dr \\
&= N \left\{\iint d(\cos\theta) d\phi\right\} \times \left(\left[r^4 (-r_0) e^{-\frac{r}{r_0}}\right]_0^{\infty} + 4\, r_0 \int_{r=0}^{\infty} r^3 e^{-\frac{r}{r_0}}\, dr\right) \\
&= N \left\{\iint d(\cos\theta) d\phi\right\} \times \left(\left[4\, r^3\, r_0^2\ e^{-\frac{r}{r_0}}\right]_0^{\infty} + 4\,.\,3\,.\, r_0 \int_{r=0}^{\infty} r^2 e^{-\frac{r}{r_0}}\, dr\right) \\
&= 12\, r_0^2 \times \left(\iiint \rho(\vec{x}) d^3\, x\right) \\
&= 12\, r_0^2
\end{aligned}
\tag{3.11}
$$

(10b) Integrating by part, on can show easily that if $\lambda > 0$

$$
\int r\, e^{-\lambda\, r} dr = \frac{1}{\lambda^2}
$$

Then quoting $u = |\vec{q}|$, we get to:

$$\begin{aligned}
\mathcal{F}(\rho(\vec{x})) \equiv \tilde{\rho}(\vec{q}) &= N \iiint d^3\, x e^{-i\vec{q}.\vec{x}} e^{-\frac{|\vec{x}|}{r_0}} \\
&= N \iiint r^2\, e^{-iu.r.\cos\theta} e^{-\frac{r}{r_0}} dr\, d(\cos\theta)\, d\phi \\
&= 2\pi\, N \int \left(\int e^{-iu.r.\cos\theta} d(\cos\theta) \right) r^2\, e^{-\frac{r}{r_0}} dr \\
&= 2\pi\, N \int \left[\frac{1}{-iu.r} e^{-iu.r.\cos\theta} \right]_{\cos\theta=-1}^{\cos\theta=1} r^2\, e^{-\frac{r}{r_0}} dr \\
&= 2\pi\, N \int_0^\infty \frac{r}{i\,u} \left(e^{-\frac{r}{r_0}+iu.r} - e^{-\frac{r}{r_0}-iu.r} \right)\, dr \\
&= N\, \frac{2\pi}{i\,u} \left(\frac{1}{\left(\frac{1}{r_0} - i\,u\right)^2} - \frac{1}{\left(\frac{1}{r_0} + i\,u\right)^2} \right) \\
&= \frac{2\pi}{i\,u} N \left(\frac{\left(\frac{1}{r_0} + i\,u\right)^2 - \left(\frac{1}{r_0} - i\,u\right)^2}{\left(\frac{1}{r_0^2} + u^2\right)^2} \right) \\
&= 2\pi\, N\, r_0^3 \frac{1}{(1 + u^2\, r_0^2)^2} \propto \left(1 + u^2\, r_0^2\right)^{-2}
\end{aligned} \tag{3.12}$$

(11) To measure $G_e(q^2)$, one can measure the differential cross section in bins of θ and then divide the measured cross -section in each θ bin by the expected cross-section from a point-like particle. Then one can get the distribution of $G_e(q^2)$ since

$$\begin{aligned}
q^2 &= (k_1 - p_1)^2 = (E - E)^2 - (\vec{p}' - \vec{p})^2 \\
q^2 &= -|(\vec{p}'||\vec{p}|(1 - \cos\theta) \\
q^2 &= -E^2(1 - \cos\theta),
\end{aligned} \tag{3.13}$$

so each θ bin corresponds to given q^2 and we can get the full $G_e(q^2)$ distribution and therefore extrapolate the derivative at $q^2 = 0$.

(12) The charge distribution in the proton follows an exponential distribution and the charge radius is $R = \sqrt{12/0.71}/1000\ \mathrm{MeV}^{-1} = 0.822\ \mathrm{fm}$