Mid-term exam of Particle Physics
Thursday November 15$^{\text{th}}$ 2018

Duration: 3 hours
6 printed pages

Allowed material: PDG booklet, simple calculator.
A formulae sheet for exercice V is provided at the end.

Solve on two separate sheets exercices I-IV and exercice V.

Approximate duration per exercice:

Ex. I:	20 min.	Ex. II:	25 min.
Ex. III:	30 min.	Ex. IV:	30 min.
Ex. V:	75 min.		

Exercice I
Questions on the lecture

Reply shortly and succinctly to the questions below. The shortest answer that details in a comprehensive manner all the relevant arguments is the best.

1. What is G-parity? Explain the logic behind this quantum number and in which cases is it useful.
2. Explain the behavior of the QCD running coupling constant. Which characteristics of the strong interaction does it allow to explain?

Exercice II
Allowed and forbidden processes, Feynman diagrammes

For each of the processes below, determine whether it is allowed or forbidden. For the forbidden processes, explain why, giving *all the possible reasons* (here we do not require to take into account multiplicative quantum numbers). For the allowed processes, specify and justify by which *dominant* interaction they occur and draw the corresponding Feynman diagrams (one per process). Note on the diagram the names of all real and virtual particles.

1. $\Delta^{++} \longrightarrow \rho(770)^+\rho(770)^+$
2. $\pi^0 n \longrightarrow K^0\Lambda$
3. $D^- \longrightarrow \tau^-\bar{\nu}_\tau$
4. $\overline{K}^0 \longrightarrow \mu^+\mu^-$
5. $K^+ \longrightarrow \pi^0\tau^+\bar{\nu}_\tau$

Exercice III

The $\omega(782)$ meson

1. In this exercice we will use the abbreviation ω to designate the $\omega(782)$ meson. Recall a few characteristics of this particle: its quark composition, baryon number, flavors, hypercharge, spin and mass. Also recall the three most probable decay modes of the ω, with their respective branching fractions (you may use the PDG booklet).

2. The PDG indicates that the full width of the ω is 8.49 MeV . What is its lifetime? (See PDG for the numerical value of $\hbar$ or $\hbar c$ in useful units.) Briefly justify the fact that it indicates that at least one of the main decay modes of the ω occurs via the strong interaction.

3. Determine the interaction via which occurs each of the main three decay modes of the ω. Give *all the relevant arguments you can find*, and explain why in the present case it is not useful to consider C and P conservation among these arguments.

4. Compare the full width (or lifetime) of the ω to that of the $\rho(770)$ and comment.

5. Determine the intrinsic parity of the ω using one of the main decay modes. The only input you can use is the spin of the ω and the quantum numbers of the daughter particles.

6. Using only the fact that pions are bosons, show that the decay mode $\omega \to \pi^0\pi^0$ is forbidden.

7. Explain why the ω is an eigenstate of charge conjugation, and obtain the corresponding eigenvalue exploiting the mode $\omega \to \pi^0\gamma$ (use as input only the quantum number of the daughter particles). Independently of the previous question, conclude that the decay mode $\omega \to \pi^0\pi^0$ is forbidden by certain interactions, specifying which.

Exercice IV

Isospin and phase space

1. The reactions:

$$(1)\;\; K^-d \to \Sigma^-p \quad ; \quad (2)\;\; K^-d \to \Sigma^0 n$$

   occur predominantly via strong interaction. We remind that the deuteron, d, is an isospin singlet. Compute the ratio of cross sections $\frac{\sigma_{(1)}}{\sigma_{(2)}}$ with the same center-of-mass energy. We will neglect here the phase-space difference between the two reactions. Explain the procedure.

2. The resonance $X(1520)$ decays via strong interaction into $n\pi^0$ and $p\pi^-$, with approximate branching fractions of 18% et 36%, respectively. What is its isospin state? Explain.

3. We remind that the cross section of a 2-body reaction verifies $d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\Phi$, where F is the flux factor, depending only on the initial state, $\mathcal{M}$ is the matrix element and $d\Phi$ the phase space factor

$$d\Phi = \frac{1}{16\pi^2}\frac{p^*}{\sqrt{s}}d\Omega\ .$$

   Here, $\sqrt{s}$ is the center-of-mass energy, p^* is the momentum of one of the final-state particles in the center-of-mass frame and $d\Omega$ the corresponding solid angle element.

   (a) Express p^* as a function of $\sqrt{s}$ and the masses of the two final-state particles: m_1, m_2.

   (b) Using the full expression of $d\sigma$, write again the ratio of cross sections for the two reactions of question 1 as a function of s (it is still the same for the two reactions) and the masses of the final-state particles. No numerical computations are required.

   (c) Comment *qualitatively* the phase space influence for $\sqrt{s}$ from threshold to infinity.

Exercice V

The proton structure

In this part, we will focus on studying the proton structure using an electron probe. We will seek to define and determine the proton radius (the typical size of a proton is $R_p \approx 1$ fm). Note that if you cannot find the answer to one of the questions below, you will still be, in general, able to continue.

1. What is the approximate minimum electron energy needed to be sensitive to the proton size? What are the corresponding β and γ? Are we in the relativistic regime? In the following, we will use such electrons to determine the proton size.

2. In this question we will study the proton induced electric field.

   (a) From $\partial_\mu F^{\mu\nu} = j^\nu$, where $j^\nu \equiv (Q\rho(x), Q\vec{j}(x))$ is the charge current density, and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, find back the Poisson equation:

   $$\nabla^2 V(x) = -Q\ \rho(x). = -Q\ \rho(\vec{x}),$$

   where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$, Q is the total charge and $\rho(x)$ the charge distribution.

   In the following, the electric field is due to a proton, which has a charge $Q = -e$, and we want to determine the charge density that will be assumed to be time-independent, *i.e.* $\rho(x) \equiv \rho(\vec{x})$. Note that $\rho(x)$ is a density distribution so it is normalized by definition: $\iiint d^3x \rho(\vec{x}) = 1$.

   (b) Using the Fourier transform of $V(x)$, $\tilde{V}(q)$, that of $\rho(x)$, $\tilde{\rho}(q)$ and the Poisson equation, show that:

   $$\tilde{V}(q) = \frac{Q}{|\vec{q}|^2}\ \tilde{\rho}(q)\,.$$

   If we consider a point-like charge, *i.e.* $\rho(q) = \delta^3(\vec{x})$, find that the Fourier transform of a Coulomb field if $\tilde{V}_{\mathrm{Clb}}$:

   $$\tilde{V}_{\mathrm{Clb}}(q) = \frac{Q}{|\vec{q}|^2}\,.$$

3. We consider the *time-independent* A^μ potential created by the proton as a classical field. Draw the Feynamn diagram associated to this interaction in the semi-classical approach. Write the corresponding matrix element (note: remember that in this case $\tilde{A}(q) = \delta(q^0)\tilde{A}(\vec{q})$ and that we do not consider $\delta(q^0)$ in the matrix element).

   Use the notations:

   - incoming electron 4-momentum $p_1 \equiv (E, 0, 0, p)$,
   - outgoing electron 4-momentum $k_1 \equiv (E', p'\sin\theta, 0, p'\cos\theta)$.
   - ξ_s the spinor basis for the incoming electron, so the corresponding Dirac bi-spinor $u_s(p)$,
   - $\eta_{s'}$ the spinor basis for the outgoing electron, so the corresponding Dirac bi-spinor $u_{s'}(p')$.
   - q the transferred momentum to the photon: $q^\mu = (k_1 - p_1)^\mu$.

   Simplify the formula of Eq. 7 for the cross section in this case, by integrating over d^3p_f with the dirac $\delta(E_f - E_i)$. What does it imply in the matrix element for E' and $p' \equiv |\vec{p}'|$?

4. Write q^2 as a function of the variables (p, θ).

5. Since we want to use relativistic electrons, we cannot use the approximation made in the Rutherford scattering formula. Let's first consider the proton as a point-like particle ($\Rightarrow$ Coulomb potential). Use the helicity basis of the outgoing and incoming electrons from Eq. 6. Forcing $E' = E$ and $p' = p$ compute the different matrix elements (without forcing $\beta = 1$) as a function of $\eta_+, \eta_-, \xi_+, \xi_-$:

   - $\mathcal{M}(+,+)$ incoming helicity+, outgoing helicity+,
   - $\mathcal{M}(+,-)$ incoming helicity+, outgoing helicity−,
   - $\mathcal{M}(-,+)$ incoming helicity−, outgoing helicity+,
   - $\mathcal{M}(-,-)$ incoming helicity−, outgoing helicity−.

6. In the limit $\beta = 1$ you should find that $\mathcal{M}(+,-) = \mathcal{M}(-,+) = 0$. Can you justify this result *without calculations* ?

7. We now need to compute the different matrix elements using the actual spinors ξ_+, ξ_- and η_+, η_-.

   (a) Express the helicity operator $h_2(p)$ and justify that $\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

   (b) Express the helicity operator $h_2(p')$ as a function θ, demonstrate that $\eta_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$, guess or find the value of η_-.

8. Using these expressions of the spinor basis, the final cross section is obtained by summing over the final spin and averaging over the initial spin, *i.e.*

$$d\sigma_{\mathrm{tot}} = \frac{1}{2}\left[d\sigma(+,+) + d\sigma(+,-) + d\sigma(-,+) + d\sigma(+,+)\right] ,$$

   where $\sigma(a,b)$ corresponds to the matrix element from (5). Show that the final cross-section is (using $Q = -Ze$ for the proton $Z = 1$):

$$\frac{d\sigma_{\mathrm{tot}}}{d\Omega} = \frac{Z^2\,\alpha^2}{4\,p^2\,\beta^2\,\sin^4\frac{\theta}{2}}\left(1 - \beta^2\,\sin^2\frac{\theta}{2}\right) . \tag{1}$$

   This is named the Mott cross section.

9. Consider that the proton has an extended spatial charge $\rho(x)$. Show that the scattering amplitude is modified as follows:

$$\frac{d\sigma_{\mathrm{tot}}}{d\Omega} = \left[\frac{d\sigma_{\mathrm{tot}}}{d\Omega}\right]_{\mathrm{Mott}} \times |G_e(q^2)|^2 . \tag{2}$$

   Express $G_e(q^2)$ as a function of $\tilde{\rho}(q)$?

10. In this case, one can define the charge radius of the proton from the spatial distribution as the RMS (root mean squared) of the ρ distribution, *i.e.*:

$$R = \sqrt{<|\vec{x}|^2>} \qquad \text{with} \qquad <|\vec{x}|^2> = \iiint |\vec{x}|^2 \rho(\vec{x}) d^3\,x .$$

   If the charge distribution is exponential, then $\rho(\vec{x}) = N \times e^{-\frac{r}{r_0}}$, where N is a normalisation factor (such that $\iiint d^3\,x\rho(\vec{x}) = 1$). Integrating by parts (note that you do not need to compute N beforehand), show that:

$$R = \sqrt{12}\,r_0 .$$

11. Show that the Fourier transform of $N \times\ e^{-\frac{r}{r_0}}$ is:

$$\tilde{\rho}(\vec{q}) \propto \left(1 - r_0^2\, q^2\right)\ ,$$

with $q^2 = -|\vec{q}|^2$.

12. From these results and equation 2, describe an experimental procedure to measure $G_e(q^2)$ and thus the size of the proton. Assume you have at your disposal an electron beam, a target of hydrogen, a small size detector which is able to detect the passage of an electron.

13. The $G_E(q^2)$ curve extracted from elastic scattering data as well as a fit to this data is given in Fig. 1. What do you conclude on the charge distribution in a proton and its average charge radius in fm ? NB: the unit in the plot is 0.71 GeV^{-2}

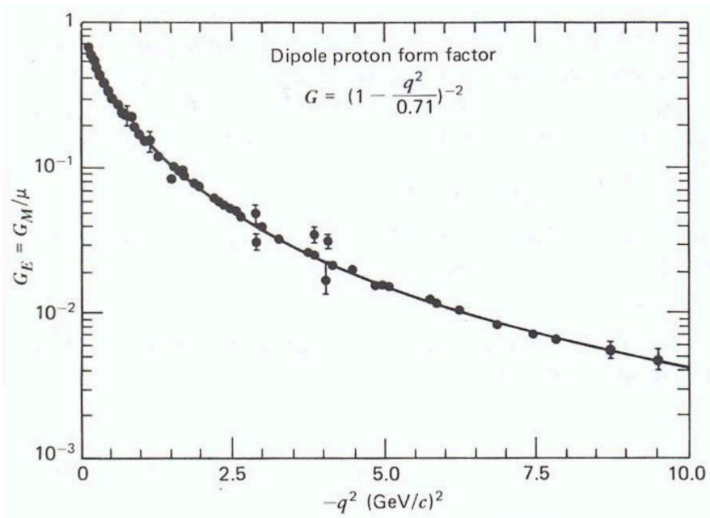


Figure 1: Proton electric form factor $G_E(q^2)$ extracted from experimental elastic scattering data.

Useful fomulae

The proton structure

γ and Pauli matrices :

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad ; \quad \vec{\sigma} \equiv \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \tag{3}$$

The bi-spinor conjugates are $\bar{u} \equiv u^\dagger \gamma^0$ and $\bar{v} \equiv v^\dagger \gamma^0$.

The relations between the Fourier transform $\tilde{f}(q)$ of a function $f(x)$ and the function itself are give by:

$$\begin{aligned} \mathcal{F}(f(x)) &\equiv \tilde{f}(q) \\ f(x) &= \int \frac{d^4\, q}{(2\pi)^4} e^{-i\, q.x} \tilde{f}(q) \\ \tilde{f}(q) &= \int d^4\, x\ e^{i\, q.x} f(x) \end{aligned} \tag{4}$$

A useful basis to express the Dirac bi-spinor is the helicity basis where $\eta_s \equiv (\eta_+, \eta_-)$, with $\eta_{+/-}$ corresponding to the helicity $+1/2$ or $-1/2$. By definition, $\eta_{+/-}$ are eigenvectors of the helicity operator

$$\begin{aligned} h_2(p) &\equiv \frac{\vec{p}.\vec{\sigma}}{|\vec{p}|} \\ h_2(p)\ \eta_+ &= +\eta_+ \\ h_2(p)\ \eta_- &= -\eta_- \end{aligned} \tag{5}$$

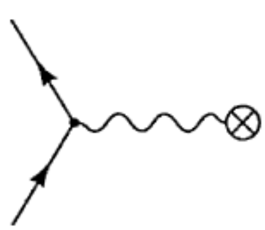
In this basis, one can write the Dirac bi-spinor:

$$u_+(p) = \sqrt{\gamma\, m} \begin{pmatrix} \sqrt{1-\beta}\ \eta_+ \\ \sqrt{1+\beta}\ \eta_+ \end{pmatrix} \quad ; \quad u_-(p) = \sqrt{\gamma\, m} \begin{pmatrix} \sqrt{1+\beta}\ \eta_- \\ \sqrt{1-\beta}\ \eta_- \end{pmatrix} \tag{6}$$

where γ and β are the corresponding β, γ of the particle, *i.e.* $p \equiv (E, \vec{p})$ such that $E = \gamma m$ and $|\vec{p}| = \gamma\beta m$.

We remind the time-independent cross section in the semi-classical approach:

$$d\sigma = \frac{1}{2\, E_i\, \beta_i} \, |\mathcal{M}(i \to f)|^2 \ (2\pi)\delta(E_f - E_i) \ \frac{d^3 p_f}{(2\pi)^3\ 2\ E_f} \tag{7}$$



$$= -ie\gamma^\mu \widetilde{A}_\mu(\mathbf{q}),$$

Figure 2: Feynman rule for the classical field (note that e is the charge of the electron)