Exercise: Expansion and Reaction rates in "natural" units

NPAC-Cosmology, December 2018

The use of "natural units" where $\hbar = c = k_B = 1$ is widespread in fundamental physics because in many problems a mass, M, determines the relevant energy scale Mc^2, length scale $\hbar c/Mc^2$, and time scale $\hbar/Mc^2$. It is clear from these three relations that the factors of $\hbar$ and c can be determined by dimensional analysis and can be omitted without creating ambiguity. For numerical results, one need only remember $\hbar c = 197 MeV\, fm$ and $c = 3 \times 10^8 m\, s^{-1}$.

In the six equations below, use dimensional analysis and a bit of physics to determine the exponents $(\alpha, \beta, \gamma....)$. Send your results to james.rich@cea.fr before noon, Thursday Dec 20 in the form of six lines containing integers, e.g.

0 1 2
1 2 3
2 3 4
3 4 5 6
4 5 6 7 8
5

1. The dimensions of Newton's constant, G, are defined by the force law: $F = Gm_1m_2/r^2$. It can be used to define the "Planck mass", $m_{\rm PL}$ through

$$G = m_{\rm PL}^{\alpha} \hbar^{\beta} c^{\gamma} \tag{1}$$

The correct values of (α, β, γ) give $m_{\rm PL}c^2 = 1.22 \times 10^{19}$GeV. The fact that it involves $\hbar$ suggests that $m_{\rm PL}c^2$ gives the energy scale where quantum gravity effects become important.

2. The low energy limit of the photon-electron scattering is called "Thompson scattering". Its cross-section, $\sigma_{\rm T}$, can be calculated using classical electromagnetic theory and therefore does not involve $\hbar$: $\sigma_{\rm T} \propto (e^2/4\pi\epsilon_0)^2 f(m_e, c)$ where f is a function to be determined. Inspite of its classicality, it is convenient to replace the electric charge with the fine-structure constant, $\alpha_{em} = e^2/4\pi\epsilon_0\hbar c \sim 1/137$ and write

$$\sigma_{\rm T} \propto\ (\alpha_{em}\hbar c)^{\alpha} m_e^{\beta} c^{\gamma} \tag{2}$$

A complete calculation gives a numerical prefactor, $8\pi/3$.

3. For cosmological temperatures much less than the the electron mass, $(k_B)T \ll m_e(c^2)$, the photon mean-free path is determined by scattering on free-electrons. To calculate the photon mean free path, we need the number density of electrons, n_e. The ratio of the number density of electrons to that of photons, n_γ, is independent of time

(after nuclear reactions stop) with $n_e/n_\gamma \equiv \eta_e \sim 5\times 10^{-10}$. We can then write the inverse of the photon mean free path as

$$\lambda_{mfp}^{-1} = n_e\sigma_T \sim \eta_e n_\gamma \sigma_T = \eta_e T^\alpha \alpha_{em}^\beta m_e^\gamma f(k_B,\hbar,c) \tag{3}$$

where f is a function of limited interest. If you have found the correct (α,β,γ), ignoring $f(k_B,\hbar,c)$ should leave a quantity with dimentions of energy when T and m_e are interpreted as energies. To obtain an inverse length, one need only divide by $\hbar c$.

4. By itself, λ_{mfp} might not mean very much to you. It is better to compare it with the Hubble length, $c/H(t)$ given by the Friedman eqn, $H^2 = 8\pi G\rho/3$. During the radiation epoch, $k_BT > 1eV$, we have $\rho \propto gT^4$, giving

$$\frac{\lambda_{mfp}}{c/H} = \sqrt{g}\eta_e^{-1}\alpha_{em}^\alpha T^\beta m_e^\gamma m_{pl}^\delta F(k_B,\hbar,c) \qquad T > T_{eq} \tag{4}$$

The correct $(\alpha,\beta,\gamma,\delta)$ should give a dimensionless quantity when $T, m_e, m_{\rm PL}$ are all interpreted as energies, allowing us to ignore $F(k_B,\hbar,c)$ (since there are no dimensionless combinations of $\hbar, c, k_B$). Furthermore, at high temperatures, $T >> 1$eV, the ratio should be $\ll 1$ implying that the photons and electrons form a fluid for the purposes of studying sound waves of wavelength much smaller than c/H.

5. After the start of matter domination, the density is a factor $\eta_{mb} \sim 5$ greater than the baryon density: $\rho = \eta_{dm}\rho_b$. The baryon density is $\rho_b = \eta m_p n_\gamma$ where m_p is the proton mass, $\eta \sim 6\times 10^{-10}$ is the baryon-photon number ratio. The ratio of the photon mean free path and the Hubble length is then given by

$$\left[\frac{\lambda_{mfp}}{c/H}\right]^2 = \eta_e^{-2}\eta_{mb}\eta\alpha_{em}^\alpha T^\beta m_e^\gamma m_{pl}^\delta m_p^\epsilon f(k_B,\hbar,c) \qquad T < T_{eq} \tag{5}$$

The correct values of $(\alpha,\beta,\gamma,\delta,\epsilon)$ should give $\lambda_{mfp}/(c/H) < 1$ at high temperature and > 1 at low temperature. Setting the ratio equal to unity defines the temperature, T_t, when the universe becomes transparent. Estimate it's order of magnitude:

$$k_BT_t \sim 10^\alpha\, eV \tag{6}$$

In fact, the universe becomes transparent before this, at $T \sim 0.2$eV where the electrons are attached to nuclei, lowering significantly the photon-matter scattering cross section.

Exercise: WIMP relic density

NPAC-Cosmology, December 2017

The purpose of this exercise is to calculate the relic density of generic "weakly interacting massive particles" (WIMPs) by numerically integrating the Boltzmann equation.

We start by studying the phase-space distributions in thermal equilibrium at temperature T. For free particles, the individual particle states are labeled by the momentum $\vec{p}$. We consider the case of equal numbers of particles and antiparticles, so the mean number of particles, $N_{\vec{p}}$, in a given state is given by the Fermi-Dirac or Bose-Einstein distribution with zero chemical potential:

$$N_{\vec{p}} = \frac{g}{\exp(E_{\vec{p}}/kT) \pm 1} \qquad E_{\vec{p}} = (p^2c^2 + m^2c^4)^{1/2}$$

where the $+$ $(-)$ is for FD (BE) statistics, g is the number of spin states, and m is the particle mass. The number of particles per unit volume, n, is found by putting the particles in a box of volume L^3, imposing periodic boundary conditions on $\vec{p}$ and then summing over $\vec{p}$:

$$n \;=\; L^{-3}\sum_{\vec{p}} N_{\vec{p}} \;\sim\; \frac{g}{(2\pi\hbar)^3}\int_0^\infty \frac{4\pi p^2 dp}{\exp(E_{\vec{p}}/kT) \pm 1}$$

Using $pdp = E_{\vec{p}}dE_{\vec{p}}$ and changing to the dimensionless variable $y = (E - mc^2)/kT$, we find

$$n \;=\; \frac{g}{2\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \int_0^\infty dy \frac{(y + mc^2/kT)\, y^{1/2}\,(y + 2mc^2/kT)^{1/2}}{\exp(mc^2/kT)\exp(y) \pm 1} \qquad (1)$$

For $m = 0$ the integral is independent of T so $n \propto T^3$ for both bosons and fermions (which we could have shown by dimensional analysis!).

1. Integrate (1) numerically for $m = 0$ to verify the Planck law $n = (g \times 1.2/\pi^2)(kT/\hbar c)^3$ for photons. For fermions verify $n = (g \times 1.2/\pi^2)(kT/\hbar c)^3 \times (3/4)$. (Python suggestion: `y=np.linspace(0.,50.,0.005)` , evaluate the integrand, then sum.)

We want to study a spin 1/2 wimp, χ, with the mass, number of spin states, and total annihilation cross section times relative velocity

$$m_\chi = 50 m_{proton} \qquad g_\chi = 2 \qquad \sigma_a v = c \times 10^{-41} m^2$$

(The value of $\sigma_a v$ was chosen because it will give a relic density near the observed density of matter.) We want to follow the number density, n_χ from $kT = 100 m_\chi c^2$, when they are relativistic, to $kT = m_\chi c^2/100$, when they are non-relativistic. To simplify the problem, we will suppose that the total energy density is dominated by a large number of relativistic degrees of freedom, $g_{tot} = 100$, so that the expansion rate is insensitive to the fact that the wimps mostly disappear between $T = 100m_\chi$ and $T = m_\chi/100$. In this case, $T \propto 1/a(t)$ and we can set up the problem in python by `a=10**np.linspace(0.,4.,4001)` and `kTmc2=100./a` where the variable `kTmc2` is $kT/m_\chi c^2$.

2. Evaluate $n_\chi(T) = n_{\chi\mathrm{eq}}(T)$ assuming the thermal equilibrium value (1).
(Suggestion: `nchithermal=np.zeros(len(a),np.float)` then evaluate `nchithermal[i]` for each `kTmc2[i]`)
Plot $n_{\chi\mathrm{eq}}/n_\gamma$ vs. T ($g_\gamma = 2$). Verify that for $T >> m_\chi$ $n_{\chi\mathrm{eq}}(T)$ approaches the Planck law and that for $T << m_\chi$ it approaches $g_\chi(m_\chi T/2\pi)^{3/2}\exp(-m_\chi/T)$. (factors of $k, \hbar, c$ to be included).

The wimp annihilation rate, Γ, is the inverse of the lifetime of wimps against annihilation with another wimp. The expansion rate, H, is the inverse of the time for density or temperature to change significantly. They are given by

$$\Gamma \;=\; \sigma_a v n_\chi \qquad\qquad H \;=\; \left(\frac{8\pi}{3}G\rho\right)^{1/2} \;\sim\; \left(\frac{8\pi}{3}G g_{tot}(\pi^2/30)\frac{(kT)^4}{(\hbar c)^3 c^2}\right)^{1/2}$$

Note that the factors of $(k, \hbar, c)$ give the dimension of $1/time$ to Γ and H.

3. Assuming the n_χ is given by the thermal equilibrium value $n_{\chi\mathrm{eq}}(T)$, plot $\Gamma_{\mathrm{eq}}(T)$. (log-log scale). Overplot $H(T)$.

If you have calculated things correctly, the annihilation and expansion rates should be equal near $T \sim m_\chi/30$. For $T < m_\chi/30$ we have $\Gamma(T) < H(T)$ and we cannot expect the annihilation rate to maintain the equilibrium value of n_χ. In this case, n_χ must be evaluated using the Boltzmann equation.

The Boltzmann equation for the phase-space density is considerably simplified if one can integrate over momentum and deal only with the total density, $n_\chi(a)$:

$$\frac{dn_\chi}{dt} \;=\; -3Hn_\chi(t) - \langle\sigma_a v\rangle n_\chi(t)^2 + \sum_i \langle\sigma_{ii\rightarrow\chi\chi} v\rangle n_i^2$$

The first term accounts for the dilution due to the expansion and by itself would give $n_\chi(t) \propto a(t)^{-3}$. The second term accounts for annihilation, $\chi\chi \rightarrow ii$ with $\sigma_a v$ being the total cross section times velocity. The third term accounts for formation of $\chi\chi$ and the sum is over all particles i that can combine to form χ. The $\langle\ \rangle$ means average over the relative velocity distribution of particle pairs, assumed to be thermal. (Unnecessary for annihilation since we assume $\sigma_a \propto 1/v$.) The cross sections for $\chi\chi \rightarrow ii$ and $ii \rightarrow \chi\chi$ are necessarily related by symmetry considerations so it is convenient to eliminate the latter by using that fact that in thermal equilibrium the annihilation and formation rates are equal: $n_{\chi\mathrm{eq}}(T)^2\langle\sigma_a v\rangle = \sum_i n_i^2\langle\sigma v\rangle_{ii}$. This gives

$$\frac{dn_\chi}{dt} \;=\; -3Hn_\chi(t) - \langle\sigma_a v\rangle(n_\chi(t)^2 - n_{\chi\mathrm{eq}}(T)^2)$$

The two terms on the right can be combined to give a form that is numerically easier to handle:

$$\frac{d(na^3)}{da^3} = \frac{-\langle\sigma_a v\rangle}{3H(a)}(n_\chi(t)^2 - n_{\chi\mathrm{eq}}(T)^2) = \frac{-\langle\sigma_a v\rangle}{3H(a)}(n_\chi(t) + n_{\chi\mathrm{eq}}(T))(n_\chi(t) - n_{\chi\mathrm{eq}}(T)) \quad (2)$$

If $n_\chi(t) > (<)\, n_{\chi\mathrm{eq}}(T)$ then $d(na^3)/da^3$ is negative (positive) which pushes the system back to equilibrium. The second form indicates that if there is a small disequilibrium, $|n_\chi(t) - n_{\chi\mathrm{eq}}(T)|/n_{\chi\mathrm{eq}}(T) \ll 1$, then equilibrium is restored after a time corresponding to $da/a \sim H/\langle\sigma_a v\rangle n_\chi(T) \sim H/\Gamma(T)$. If $H/\Gamma > 1$ then $da/a > 1$ and equilibrium cannot be maintained because the temperature drops faster than annihilations can reduce $n_\chi a^3$.

The Boltzmann equation (2) can be solved by integrating step by step starting at a sufficiently high temperature where $n_\chi(a_{start}) = n_{\chi\mathrm{eq}}(a_{start})$. At each new value of a, the right hand side of (2) is calculated and used to calculate the value of $n_\chi a^3$ at the new a.

5. Starting at a value of a where $\Gamma/H \sim 100$, integrate the Boltzmann equation to find $n_\chi(a)$. Plot $n_\chi(a)/n_\gamma(a)$ and $n_{\chi\mathrm{eq}}(a)/n_\gamma(a)$. ($g_\gamma = 2$)

At the last point on your integration, a_{lp}, you should have $n_{\chi\mathrm{eq}} << n_\chi$, in which case the Boltzmann equation can be integrated exactly. Integrating from a_{lp} to $a \to \infty$ you can show

$$1 - \frac{(n_\chi a^3)_\infty}{(n_\chi a^3)_{lp}} = \left(\frac{\langle\sigma_a v\rangle n_\chi}{H}\right)_{lp}$$

i.e. the remaining annihilations correspond to the annihilation rate at the last point times the Hubble time at the last point.

6. Verify that the remaining annhilations change only slightly $n_\chi a^3$.

To calculate the present-day density of χ, we assume that the expansion after the last point is adiabatic so that sa^3, like $n_\chi a^3$, is constant. The ratio n_χ/s is therefore time-independent.

7. Calculate s at the last point of the Boltzmann integration assuming the entropy is dominated by fermions with $g_{tot} = 100$. Calculate s today assuming ($T_\gamma = 2.7K, g_\gamma = 2$) and ($T_\nu = 2K, g_\nu = 6$). Calculate n_χ today. Calculate ρ_χ today ($m_\chi = 50 m_{proton}$) and compare with the critical density, $\rho_c \sim 10^{-26} kg\, m^{-3}$.