# Master NPAC

**Cosmology – extra homework**

January 20, 2020

## 1 Quintessence

Some cosmologists speculate that the universe may contain a quantum field called "*quintessence*" (from *quinta essentia*, the fifth element also named *aether* in Aristotle world-view). This quantum field has a positive energy density and a negative equation-of-state parameter $w_Q < 0$.

Let suppose that we live in a spatially flat universe, containing only matter ($\Omega_{m,0} \leq 1$) and quintessence with $w_Q = -1/2$ and $\Omega_{Q,0} = 1 - \Omega_{m,0}$.

1. At what scale $a_{mQ}$ will the energy density of quintessence and matter be equal. Express it as a function of $\Omega_{m,0}$ and $\Omega_{Q,0}$.
2. What is $a(t)$ when $a \ll a_{mQ}$? and when $a \gg a_{mQ}$?
3. Solve the Friedmann equation to find $a(t)$ for this universe (without approximation).

   *Hints:*
   $$\int \frac{x^{1/2}\mathrm{d}x}{\sqrt{1+\beta x^{3/2}}} = \frac{4}{3\beta}\sqrt{1+\beta x^{3/2}}$$
4. What is the current age of this universe, as a function of $H_0$ and $\Omega_{m,0}$?
5. Describe the properties of a universe entirely made of quintessence ($\Omega_m \approx 0$). What would be the current age of this universe, and the current particle horizon distance ?

## 2 Closed Universe

Consider a closed universe which contains only matter : $\Omega_0 = \Omega_{m,0} > 1$.

1. Describe briefly the properties of such a universe, its curvature, and its dynamics (a drawing may help).

2. Write the Friedmann equation for this universe. Compute the value $a_{\max}$ of the scaling factor $a$ at maximum expansion.

3. Show that $H_0$, $\Omega_0$ and the current universe curvature radius $a_0$ are linked by:

$$a_0 = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_0 - 1}}$$

4. Calculate the horizon distance $d_{\text{hor}}(t) = a_0 \chi_{\text{hor}}(t)$ as a function of time. Remember that the comoving coordinate $\chi_{\text{hor}}$ of the horizon at time $t$ is defined by:

$$\chi_{\text{hor}}(t) = \int_0^{a(t)} \frac{dt'}{a_0(t')}$$

Show that at the moment of maximum expansion ($a = a_{\max}$, $t = t(a_{\max})$), the horizon includes the entire universe, *i.e.*:

$$\chi_{\text{hor}}(a_{\max}) = \pi.$$

*Hints:*

$$\int_A^B \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \Big[\arcsin(2x-1)\Big]_A^B \qquad x = \frac{\Omega_0 - 1}{\Omega_0} \times a$$

5. Verify that the evolution of the universe may be described by the following parametric equations:

$$a(\eta) = A\,(1 - \cos\eta)$$
$$t(\eta) = B\,(\eta - \sin\eta)$$

Give the expression of $A$ and $B$ as functions of $H_0$ and $\Omega_0$. What is the value of $\eta$ at maximum expansion? Describe briefly the resulting dynamics.

6. Show that the age of the universe at maximal expansion is:

$$t(a_{\max}) = \frac{\pi}{2} \frac{1}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$$

7. At some time $t_1 > t(a_{\max})$ during the contraction phase of this universe, an astronomer named Edwin Elbbuh discovers that all nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance ; he measures as well $H_1 < 0$ and $\Omega_1 > 1$. Knowing $H_1 < 0$ and $\Omega_1$, how much time remains between $t_1$ and the final Big Crunch at $t = t_{\text{crunch}}$? What is the minimum blueshift our astronomer may observe?

In the 18th century, Olbers and Chéseaux noted that if the universe is homogeneous in space *and* in time, the integrated starlight must be infinite, making the night as bright as the day. Their reasoning was quite simple: if $n$ is the number of stars per unit volume and $L$ their mean luminosity then the flux we receive from the stars in a shell of thickness $dR$ at a distance $R$ is

$$df = \frac{n \times L \times 4\pi R^2 dR}{4\pi R^2} \tag{1}$$

Integrating this expression from $R = 0$ to $R = \infty$ gives an infinite flux. This is Olbers' paradox and the following exercise shows how modern cosmology resolves it.

Some useful information:

The scale parameter, $a(t)$, is determined by the Friedmann equation (neglecting radiation and assuming a flat universe):

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_\Lambda + \Omega_M(a/a_0)^{-3}\right] \qquad \Omega_\Lambda + \Omega_M = 1 \qquad a_0 \equiv a(t_0) \tag{2}$$

where $H_0$ is the present value of the expansion rate

$$\left[\frac{\dot{a}}{a}\right]_{t_0} = H_0 \qquad H_0^{-1} \sim 1.4 \times 10^{10} yr \sim 4.4 \times 10^{17} sec \qquad cH_0^{-1} \sim 4300\, Mpc \sim 1.3 \times 10^{26} m \tag{3}$$

and where $\Omega_M$ and $\Omega_\Lambda$ are determined by the present matter and vacuum densities and $H_0$:

$$\Omega_M \equiv \frac{\rho_M(t_0)}{3H_0^2/(8\pi G)} \qquad \Omega_\Lambda \equiv \frac{\rho_V}{3H_0^2/(8\pi G)} \tag{4}$$

A modern statement of Olbers' paradox talks of galaxies rather than stars. The number of galaxies per unit volume in the universe, $n(t)$, evolves with time in a way that reflects the history of galaxy formation. However, in recent times $n(t)$ has a simple form since most galaxies were formed in the distant past implying that the total number of galaxies is time-independent. Throughout this examination we will assume that this is true.

**1. Give an expression for $n(t)$ in terms of its present value, $n_0 = n(t_0)$ and $a(t)/a_0$.**

Consider the space between $\chi$ and $\chi + d\chi$ in in the solid angle on the sky $d\Omega$. At the time $t$ the volume of this space (i.e. in $Mpc^3$) is given by $a(t)^3 d\Omega \chi^2 d\chi$.

**2. How many galaxies are there in this volume at time $t$ as a function of $n_0$?**

Photons are emitted from galaxies at $(\chi_1, t_1)$ and received by us at $(\chi = 0, t_0)$. The present distance, $a_0\chi_1$, is related to the redshift, $z$, of the galaxies by

$$a_0\chi_1 = (c/H_0)\int_{1/(1+z)}^{1} \frac{d\hat{a}}{\hat{a}^2[\Omega_M \hat{a}^{-3} + \Omega_\Lambda]^{1/2}} \tag{5}$$

For $(\Omega_M, \Omega_\Lambda) = (0, 1)$ this gives

$$a_0\chi_1 = (c/H_0)z \qquad \Rightarrow z(\chi_1) = \frac{a_0\chi_1}{c/H_0} \tag{6}$$

while for $(\Omega_M, \Omega_\Lambda) = (1, 0)$

$$a_0\chi_1 = 2(c/H_0)\left[1 - \frac{1}{\sqrt{1+z}}\right] \qquad \Rightarrow 1 + z(\chi_1) = \left(1 - \frac{a_0\chi_1}{2c/H_0}\right)^{-2} \tag{7}$$

We denote by $L(t)$ the mean galactic luminosity (energy per unit time) at time $t$.

**3. What is the energy flux, $df$, (energy per unit area per unit time) that we now receive from the galaxies in $(d\chi, d\Omega)$.**

The mean galactic luminosity $L(t)$ is a complicated function that is not at all known at early times. We will assume that it has the simple form

$$L(t) = L(t_0) \quad t > t_g \qquad\qquad L(t) = 0 \quad t < t_g \tag{8}$$

where $t_g$ can be interpreted as the time when all galaxies were "turned on". At this time the scale parameter had a value $a(t_g)$ corresponding to a redshift $z_g = a_0/a(t_g) - 1$.

**4. For $(\Omega_M, \Omega_\Lambda) = (0, 1)$, find an expression for the the total energy flux $f(t_0)$ that we receive at $(\chi, t) = (0, t_0)$. The expression should be in the form**

$$f(t_0) \; = F(t_0) \int_0^{u_{max}} g(u) du \qquad\qquad u = \frac{a_0 \chi}{c/H_0} \tag{9}$$

**where $F(t_0)$ is a function of the dimensioned parameters of the problem $(c, H_0, n(t_0), L(t_0), a_0.....)$ and $g(u)$ is a dimensionless function of the dimensionless variable $u$. Specify the function $g(u)$ and $u_{max}$ but do not bother to evaluate the integral.**

**5. Same as for 4 except for $(\Omega_M, \Omega_\Lambda) = (1, 0)$.**

**6. Evaluate the integrals in 4 and 5 in the limit of very early galaxy formation: $a(t_g)/a_0 \to 0$.**

The observed galaxy density is $n_0 \sim 10^{-2} Mpc^{-3}$ and the mean galactic luminosity is $L \sim 10^{10} L_\odot$ where $L_\odot$ is the solar luminosity.

**7. Compare the order of magnitude of the flux from 6 to that we receive from the Sun at our distance $\sim 5 \times 10^{-6} pc$.**

**8. Compare the order of magnitude of the flux from 6 to that from stars in the Milky Way (total luminosity $\sim 10^{10} L_\odot$). Most of this light comes for the Milky Way's center at a distance of $\sim 10 kpc$.**

If this examination had been held at some other time $t_0' \neq t_0$, the flux given by equation 9 would be different. The most important differences would be in the dimensioned factor $F(t)$.

**9. Give $F(t_0')$ as a function of $F(t_0)$ and of $a(t_0')/a(t_0)$ for $(\Omega_M, \Omega_\Lambda) = (0, 1)$**

**10. Same as 9 but for $(\Omega_M, \Omega_\Lambda) = (1, 0)$**

The function $g(u)$ in equation 9 would also be different at $t_0' \neq t_0$ since it depends on the $\Omega_M$ and $\Omega_\Lambda$ that would be defined by replacing $t_0$ by $t_0'$ in equations 4 and 3.

**11. Give $\Omega_M(t_0')$ and $\Omega_\Lambda(t_0')$ as a function of $\Omega_M$ and $\Omega_\Lambda$ (their values at $t_0$) and of $a(t_0')/a_0$**

If you have done (11) correctly you have discovered $\Omega_M$, and $\Omega_\Lambda$ are time-independent if $(\Omega_M, \Omega_\Lambda) = (1, 0)$ or $(\Omega_M, \Omega_\Lambda) = (0, 1)$. In these two cases, the integral in equation 9 is time-independent (at least for $a(t_g)/a_0 \to 0$). This means that the time dependence of $f(t)$ is that given by $F(t)$ found in 9 and 10.

An active area of research is to determine when and how the first galaxies were formed. Figure 1 shows one of the highest redshift galaxies observed in the so-called Hubble Ultra-Deep Field. Four zooms are shown at the bottom, each zoom being an image taken with a filter centered on the marked wavelengths. The galaxy in question is seen in three of the four zooms (pointed to by the arrow).

We note that most light produced in any galaxy at $\lambda < 100nm$ is absorbed before it leaves the galaxy through photo-ionization of galactic gas.

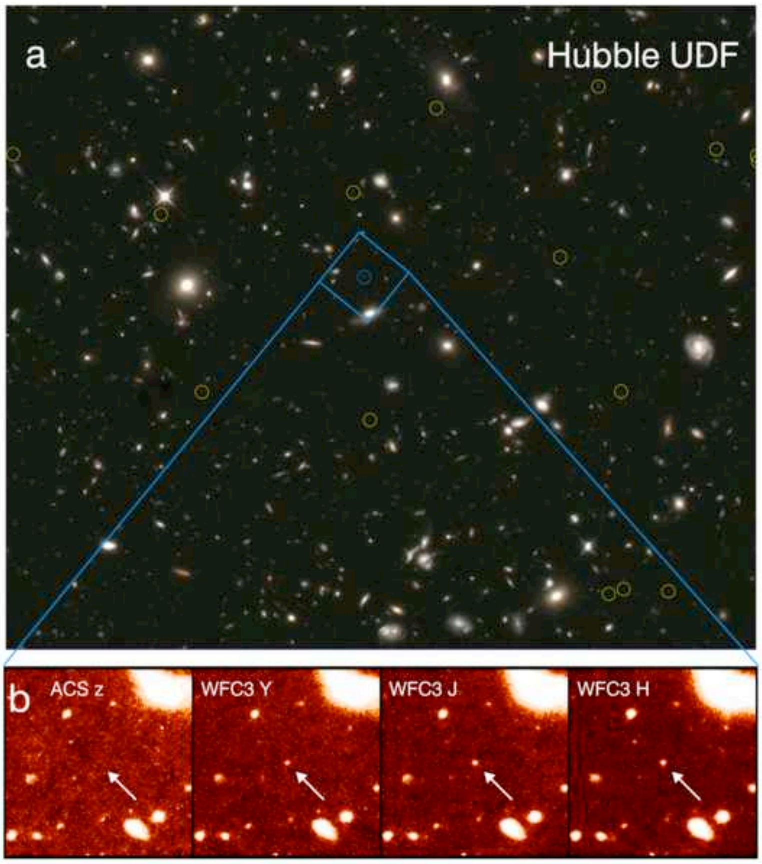
**12. Estimate the redshift of the galaxy pointed to in figure 1.**

Most stellar energy is produced in nuclear reactions that transform four protons into a helium nucleus in stellar cores. This transformation liberates $\sim 7MeV$ per proton. Most of this energy is thermalized in the star and leaves the stellar surface as starlight.

The Milky Way contains about $10^{67}$ protons ($10^{10} M_\odot$) and has a luminosity of $\sim 10^{10} L_\odot \sim 2 \times 10^{55} eV\, s^{-1}$

**13. How long can the Milky Way continue to produce starlight at the present rate before all its hydrogen has been converted to helium?**

Robertson, B. E., Ellis, R. S., Dunlop, J. S., McLure, R. J. and Stark, D. P., *Nature* **468,** 55 (2010)




~$10^{-7}$ of sky

870nm 1050nm 1200nm 1600nm