

Master NPAC

Cosmology – extra homework

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1 Quintessence

Some cosmologists speculate that the universe may contain a quantum field called “*quintessence*” (from *quinta essentia*, the fifth element also named *aether* in Aristotle world-view). This quantum field has a positive energy density and a negative equation-of-state parameter $w_Q < 0$.

Let suppose that we live in a spatially flat universe, containing only matter ($\Omega_{m,0} \leq 1$) and quintessence with $w_Q = -1/2$ and $\Omega_{Q,0} = 1 - \Omega_{m,0}$.

1. At what scale a_{mQ} will the energy density of quintessence and matter be equal. Express it as a function of $\Omega_{m,0}$ and $\Omega_{Q,0}$.

The density of a fluid of equation of state $p = w\rho$ (w constant) varies like

$$\rho \propto \left(\frac{a_0}{a}\right)^{3(1+w)} \quad (1)$$

The equality between matter and quintessence (with $w = -1/2$) is therefore reached for:

$$\rho_{m,0} \left(\frac{a_0}{a}\right)^3 = \rho_{Q,0} \left(\frac{a_0}{a}\right)^{3/2} \quad (2)$$

hence,

$$a = a_0 \left(\frac{\Omega_m}{\Omega_Q}\right)^{2/3} \quad (3)$$

For a flat universe:

$$a = a_0 \left(\frac{\Omega_m}{1 - \Omega_m}\right)^{2/3} \quad (4)$$

2. What is $a(t)$ when $a \ll a_{mQ}$? and when $a \gg a_{mQ}$?

The evolution of $a(t)$ can be obtained from the Friedmann equation. Let's rewrite it as:

$$\left(\frac{1}{x} \frac{dx}{dt}\right)^2 = H_0^2 \left(\Omega_m x^{-3} + \Omega_Q x^{-3/2} + \underbrace{(1 - \Omega_m - \Omega_Q)}_{=0} \right) \quad (5)$$

with $x = a/a_0$. We have:

$$\int_0^t dt = \frac{1}{H_0} \int_0^{a/a_0} \frac{dx}{(\Omega_m x^{-1} + \Omega_Q x^{1/2})^{1/2}} \quad (6)$$

When $a \ll a_{mQ}$ matter dominates, and the integration of the equation above gives:

$$a(t) \propto t^{2/3} \quad (7)$$

more exactly, we have:

$$\frac{a}{a_0} = \left(\frac{3}{2} H_0 \sqrt{\Omega_m} t \right)^{2/3} \quad (8)$$

When $a \gg a_{mQ}$ quintessence dominates, and we have, from the same integral:

$$a(t) \propto t^{4/3} \quad (9)$$

3. Solve the Friedmann equation to find $a(t)$ for this universe (without approximation).

Hints:

$$\int \frac{x^{1/2} dx}{\sqrt{1 + \beta x^{3/2}}} = \frac{4}{3\beta} \sqrt{1 + \beta x^{3/2}}$$

This is just working out the integral above. We have

$$t = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_m}} \int_0^{a/a_0} \frac{x^{1/2} dx}{(1 + \frac{\Omega_Q}{\Omega_m} x^{3/2})^{1/2}} \quad (10)$$

Using the hint above, we get:

$$t = \frac{4}{3} \frac{1}{H_0} \frac{\sqrt{\Omega_m}}{\Omega_Q} \left(\sqrt{1 + \frac{\Omega_Q}{\Omega_m} \left(\frac{a}{a_0}\right)^{3/2}} - 1 \right) \quad (11)$$

Inverting the equation gives:

$$a(t) = a_0 \left(\frac{\Omega_m}{\Omega_Q} \left(\frac{3}{4} H_0 \frac{\Omega_Q}{\sqrt{\Omega_m}} t + 1 \right)^2 - 1 \right)^{2/3} \quad (12)$$

4. What is the current age of this universe, as a function of H_0 and $\Omega_{m,0}$?

The current age of the Universe is again given by the equation above:

$$t = \frac{4}{3} \frac{1}{H_0} \frac{\sqrt{\Omega_m}}{1 - \Omega_m} \left(\sqrt{1 + \frac{1 - \Omega_m}{\Omega_m}} - 1 \right) \quad (13)$$

5. Describe the properties of a universe entirely made of quintessence ($\Omega_m \approx 0$). What would be the current age of this universe, and the current particle horizon distance ?

We have $w < -1/3$. The second Friedmann equation tells us the expansion is accelerating ($\ddot{a} > 0$).

Let's check that by integrating the Friedmann equation, we find:

$$t = \frac{1}{H_0} \int_0^{a/a_0} \frac{x^{-1/4}}{\Omega_Q^{1/2}} dx = \frac{4}{3} \frac{1}{H_0} \frac{1}{\sqrt{\Omega_Q}} \left(\frac{a}{a_0} \right)^{3/4} \quad (14)$$

This gives:

$$a \propto t^{4/3} \quad (15)$$

We find indeed an accelerated expansion.

We find a current age of the Universe of $\frac{4}{3} \frac{1}{H_0}$ (taking the limit $\Omega_m \rightarrow 0$).

Finally, we find the horizon distance diverges. The horizon encompasses all the universe.

2 Closed Universe

Consider a closed universe which contains only matter : $\Omega_0 = \Omega_{m,0} > 1$.

1. Describe briefly the properties of such a universe, its curvature, and its dynamics (a drawing may help).

A universe that contains only non-relativistic matter with a density above the critical density is closed and has positive curvature ($k = +1$). Its expansion reaches a maximum (a_{max}) and is followed by a contraction phase that ends in a Big Crunch event.

2. Write the Friedmann equation for this universe. Compute the value a_{max} of the scaling factor a at maximum expansion.

The first Friedmann equation is:

$$H^2(t) = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (16)$$

It can be rewritten as:

$$H^2 = H_0^2 (\Omega_0 a^{-3} + (1 - \Omega_0) a^{-2}) \quad (17)$$

At maximum expansion, $\dot{a} = 0$, $H = 0$, which gives

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1} \quad (18)$$

3. Show that H_0 , Ω_0 and the current universe curvature radius a_0 are linked by:

$$a_0 = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_0 - 1}}$$

From (1) the first Friedmann equation and (2) the definition of the critical density ($\rho_c = \frac{3H^2}{8\pi G}$), we have:

$$H_0^2 = H_0^2 \Omega_0 - \frac{k}{a_0^2} \quad (19)$$

Factorizing out H_0^2 , this gives

$$\Omega_0 - \frac{k}{a_0^2 H_0^2} = 1 \quad (20)$$

from which we get:

$$a_0 = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_0 - 1}} \quad (21)$$

4. Calculate the horizon distance $d_{\text{hor}}(t) = a_0 \chi_{\text{hor}}(t)$ as a function of time. Remember that the comoving coordinate χ_{hor} of the horizon at time t is defined by:

$$\chi_{\text{hor}}(t) = \int_0^t \frac{dt'}{a(t')}$$

The horizon distance is given by:

$$d_{\text{hor}} = a_0 \int_0^{t_0} \frac{dt}{a(t)} = a_0 \int_0^{a_0} \frac{da}{\dot{a}a} = a_0 \int_0^{a_0} \frac{da}{Ha^2} \quad (22)$$

Injecting the Friedmann equation ($H(a) = \dots$) and changing variables: $x = a/a_0$, we obtain:

$$d_{\text{hor}} = \frac{1}{H_0} \int_0^{a/a_0} \frac{dx}{(\Omega_0 x - (\Omega_0 - 1)x^2)^{1/2}} \quad (23)$$

If we pose $u = \frac{\Omega_0 - 1}{\Omega_0} x$, the integral may be rewritten:

$$d_{\text{hor}} = \frac{1}{\sqrt{\Omega_0 - 1}} \frac{1}{H_0} \times \int_0^{\frac{\Omega_0 - 1}{\Omega_0} \frac{a}{a_0}} \frac{du}{\sqrt{u(1 - u)}} \quad (24)$$

which gives:

$$d_{\text{hor}}(t) = \frac{1}{\sqrt{\Omega_0 - 1}} \frac{1}{H_0} \times \arcsin \left(2 \frac{\Omega_0 - 1}{\Omega_0} \frac{a(t)}{a_0} - 1 \right) \quad (25)$$

Using the results of the previous question, we can write:

$$\chi_{\text{hor}}(t) = \frac{1}{a_0} d_{\text{hor}}(t) = \arcsin \left(2 \frac{\Omega_0 - 1}{\Omega_0} \frac{a(t)}{a_0} - 1 \right) \quad (26)$$

Show that at the moment of maximum expansion ($a = a_{\text{max}}$, $t = t(a_{\text{max}})$), the horizon includes the entire universe, i.e.:

$$\chi_{\text{hor}}(a_{\text{max}}) = \pi.$$

At the moment of maximum expansion, $a_{\text{max}} = \frac{\Omega_0}{\Omega_0 - 1}$, hence

$$\chi_{\text{hor}} = \arcsin(2 - 1) = \pi \quad (27)$$

The horizon includes the entire Universe.

Hints:

$$\int_A^B \frac{dx}{\sqrt{x(1-x)}} = \left[\arcsin(2x - 1) \right]_A^B \quad x = \frac{\Omega_0 - 1}{\Omega_0} \times a$$

5. Verify that the evolution of the universe may be described by the following parametric equations:

$$\begin{aligned} a(\eta) &= A(1 - \cos \eta) \\ t(\eta) &= B(\eta - \sin \eta) \end{aligned}$$

Give the expression of A and B as functions of H_0 and Ω_0 . What is the value of η at maximum expansion? Describe briefly the resulting dynamics.

By differentiating a and t , forming \dot{a}/a and indentifying with the Friedmann equation terms, we obtain:

$$A = \frac{1}{2} \frac{\Omega_0}{\Omega_0 - 1} \quad B = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (28)$$

6. Show that the age of the universe at maximal expansion is:

$$t(a_{\max}) = \frac{\pi}{2} \frac{1}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$$

From the previous question, the maximum of a is reached for $\eta = \pi$. Therefore, we have

$$t(a_{\max}) = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (29)$$

7. At some time $t_1 > t(a_{\max})$ during the contraction phase of this universe, an astronomer named Edwin Elbbuh discovers that all nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance ; he measures as well $H_1 < 0$ and $\Omega_1 > 1$. Knowing $H_1 < 0$ and Ω_1 , how much time remains between t_1 and the final Big Crunch at $t = t_{\text{crunch}}$?

The function that describes the expansion is symmetric around a_{\max} . The remaining time before the big crunch is equal to the age of the universe when it was expanding, with $H_0 = -H_1$.

$$t = -\frac{2}{3} \frac{1}{H_1} \frac{1}{\sqrt{\Omega_1}} \quad (30)$$