

# Master NPAC

## Cosmology homework #2

November 28, 2019

Answers are to be sent by email to `nicolas.regnault-AT-lpnhe.in2p3.fr` by Friday Dec 5th, 2019. Please use the following format (given as an example):

<your name>

1. A
2. B
3. C
4. D
5. A
6. B
7. A

We have seen that, in a homogeneous, isotropic Universe in expansion, the metric, in comoving coordinates  $(t, \chi, \theta, \phi)$  can be written as:

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a^2(t) (d\chi^2 + r^2(\chi) (d\theta^2 + \sin^2\theta d\phi^2)) \quad (1)$$

General Relativity tells us that free particles follow geodesics  $(x^\mu(\lambda))$ , which are solution of the geodesics equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \quad (2)$$

where the  $\Gamma$  coefficients are the christoffel connections. For the metric above, we remind you that:

$$\Gamma_{00}^0 = \Gamma_{00}^i = \Gamma_{0i}^0 = 0 \quad \Gamma_{0j}^i = \frac{\dot{a}}{a} \delta_j^i \quad \Gamma_{ij}^0 = a\dot{a}\tilde{g}_{ij} \quad \Gamma_{11}^1 = \Gamma_{\chi\chi}^x = 0 \quad (3)$$

In this homework, we study the motion of a spacecraft that leaves our Galaxy ( $\chi = 0$ ) today ( $t = t_0$ ), with an initial velocity  $\frac{d\ell_{\text{phys}}}{dt} = \frac{da(t)\chi(t)}{dt} = v_0$ . We assume that all its fuel is used to generate the spacecraft's initial momentum, and that it follows a purely radial trajectory:  $(t(\tau), \chi(\tau), \theta = \text{const}, \phi = \text{const})$ .

1. Show that

$$\dot{\chi}(t_0) = \frac{d\chi}{dt}(t_0) = \frac{v_0}{a_0} \quad (4)$$

Use  $-c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$  to express  $d\tau/dt$  as a function of  $a$  and  $\chi$ :

(A)  $\frac{d\tau}{dt} = \left(1 - \frac{a^2 \dot{\chi}^2}{c^2}\right)^{1/2}$

(B)  $\frac{d\tau}{dt} = \left(1 - a^2 \chi^2\right)^{1/2}$

(C)  $\frac{d\tau}{dt} = \left(1 - a\chi/c\right)^{1/2}$

(D) None of the above

2. Express the initial values (at  $t_0$ ) of  $dt/d\tau$  and  $d\chi/d\tau$

(A)  $\frac{dt}{d\tau}|_{t_0} = 1 \quad \frac{d\chi}{d\tau} = v_0/a_0$

(B)  $\frac{dt}{d\tau}|_{t_0} = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad \frac{d\chi}{d\tau} = \frac{v_0}{a_0} \frac{1}{\sqrt{1 - v_0^2/c^2}}$

(C)  $\frac{dt}{d\tau}|_{t_0} = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad \frac{d\chi}{d\tau} = \frac{v_0}{a_0} \sqrt{1 - v_0^2/c^2}$

(D) None of the above

3. Derive from the geodesic equation obeyed by the spacecraft trajectory the differential equations followed by  $t(\tau)$  and  $\chi(\tau)$ :

(A)  $\frac{d^2 t}{d\tau^2} = -a\dot{a} \left(\frac{d\chi}{d\tau}\right)^2 \quad \frac{d}{d\tau} \left(a^2 \frac{d\chi}{d\tau}\right) = 0$

(B)  $\frac{d^2 t}{d\tau^2} = 1 \quad \frac{d}{d\tau} \left(\frac{da}{d\tau} \frac{d\chi}{d\tau}\right) = 0$

(C)  $\frac{d^2 t}{d\tau^2} = -a\dot{a}\dot{\chi}^2 \quad \frac{d}{d\tau} \left(a^2 \frac{d\chi}{d\tau}\right) = 0$

(D) None of the above

4. Derive from the above one single differential equation  $d\chi/dt = f(a, \chi, t)$ . Write the limits of this equation for a slowly moving spacecraft ( $v_0 \rightarrow 0$ ) and a spacecraft moving at relativistic speeds:  $v_0 \rightarrow c$ .

(A)  $(v_0 \rightarrow 0) \dot{\chi} = \frac{a_0 v_0}{a^2(t)} \quad (v_0 \rightarrow c) \dot{\chi} = \frac{c}{a(t)}$

(B)  $(v_0 \rightarrow 0) \dot{\chi} = \frac{v_0}{a(t)} \quad (v_0 \rightarrow c) \dot{\chi} = \frac{c}{a(t)}$

(C)  $(v_0 \rightarrow 0) \dot{\chi} = v_0 \dot{a} \quad (v_0 \rightarrow c) \dot{\chi} = \frac{a_0}{a^2(t)}$

(D) None of the above

**5.** We assume that we live in a matter-dominated Universe:  $a(t) = a_0(t/t_0)^{2/3}$ , with  $t_0 = (2/3)H_0^{-1}$ . The spacecraft reaches a galaxy at  $t = t_1$ . What is the comoving distance  $a_0\chi_1$  to this galaxy in the two cases above ( $v_0 \rightarrow 0$  and  $v_0 \rightarrow c$ ) ?

(A)  $(v_0 \rightarrow 0) a_0\chi_1 = \frac{2}{H_0} \left( 1 - \left( \frac{t_0}{t_1} \right)^{2/3} \right) \quad (v_0 \rightarrow c) a_0\chi_1 = \frac{2c}{H_0} \left( \left( \frac{t_0}{t_1} \right)^{2/3} - 1 \right)$

(B)  $(v_0 \rightarrow 0) a_0\chi_1 = \frac{2a_0}{H_0} \left( 1 - \left( \frac{t_0}{t_1} \right)^{1/3} \right) \quad (v_0 \rightarrow c) a_0\chi_1 = \frac{2a_0}{H_0} \left( \left( \frac{t_1}{t_0} \right)^{2/3} - 1 \right)$

(C)  $(v_0 \rightarrow 0) a_0\chi_1 = \frac{2v_0}{H_0} \left( 1 - \left( \frac{t_0}{t_1} \right)^{1/3} \right) \quad (v_0 \rightarrow c) a_0\chi_1 = \frac{2c}{H_0} \left( \left( \frac{t_1}{t_0} \right)^{1/3} - 1 \right)$

(D) None of the above

**6.** Same question for a vacuum-energy-dominated Universe:  $a(t) = a_0 \exp H_0(t - t_0)$

(A)  $(v_0 \rightarrow 0) a_0\chi_1 = \frac{v_0}{2H_0} (1 - e^{-2H_0(t_1-t_0)}) \quad (v_0 \rightarrow c) a_0\chi_1 = \frac{c}{H_0} (1 - e^{-H_0(t_1-t_0)})$

(B)  $(v_0 \rightarrow 0) a_0\chi_1 = \frac{v_0}{H_0} (1 - e^{-H_0(t_1-t_0)}) \quad (v_0 \rightarrow c) a_0\chi_1 = \frac{c}{2H_0} (1 - e^{-2H_0(t_1-t_0)})$

(C)  $(v_0 \rightarrow 0) a_0\chi_1 = \frac{c}{H_0} (1 - e^{-2H_0(t_1-t_0)}) \quad (v_0 \rightarrow c) a_0\chi_1 = \frac{v_0}{H_0} (1 - e^{-H_0(t_1-t_0)})$

(D) None of the above

**7.** For  $t \rightarrow \infty$  and  $v_0 \rightarrow c$  is there a limit to the distance we can travel in (1) a matter-dominated Universe and (2) a vacuum-dominated Universe ?

(A) Yes Yes

(B) Yes No

(C) No Yes

(D) No No