Problem 1

a) Distance to the galaxy

$$a_0 \chi = a_0 \frac{1}{a_0} \int_0^z \frac{c\,dz}{H(z)} = \int_0^z \frac{c\,dz}{H_0} = \frac{cz}{H_0}$$

b) Distance to the galaxy when photons were emitted

$$a_1 \chi = \frac{a_1}{a_0} \int_0^z \frac{c\,dz}{H_0} = \frac{1}{1+z} \frac{cz}{H_0}$$

c) Photon flight time

$$dt = \frac{da}{\dot{a}} \Rightarrow t = \int_{a_1}^{a_0} \frac{da}{\dot{a}} = \int_{a_1}^{a_0} \frac{1}{a} \frac{da}{H_0}$$

$$= \frac{1}{H_0} \ln\left(\frac{a_0}{a_1}\right) = \frac{1}{H_0} \ln(1+z)$$

d) area of the sphere

$$S = 4\pi a_0^2 \chi^2 \quad \text{(flat Universe)}$$

e) detector ~~of~~ area A

$$\frac{A}{S} = \frac{A}{4\pi a_0^2 \chi^2} \Rightarrow N_{det} = \frac{A N_{emitted}}{4\pi a_0^2 \chi^2}$$

f) $\Omega_k = 1 - \Omega_{tot} > 0 \Rightarrow$ Open universe

$$d = a_0 \sinh\chi > a\chi$$

$\Rightarrow$ fewer photons

## Problem 2 : Redshift drift

(a) $$z(t_0) = \frac{a(t_0)}{a(t_1)}$$

$$z(t_0+\delta t_0) = \frac{a(t_0+\delta t_0)}{a(t_1+\delta t_1)}$$

(b) $$\delta z = z(t_0+\delta t_0) - z(t_0)$$

$$= \frac{a(t_0+\delta t_0)}{a(t_1+\delta t_1)} - \frac{a(t_0)}{a(t_1)}$$

$$= \frac{a(t_0) + \dot{a}(t_0)\,\delta t_0}{a(t_1) + \dot{a}(t_1)\,\delta t_1} - \frac{a(t_0)}{a(t_1)}$$

$$\boxed{\delta t = \left(\frac{a_0}{a_1}\right)\left(\frac{1 + H_0\,\delta t_0}{1 + H_1\,\delta t_1} - 1\right)}$$

(c) $$\chi = \frac{1}{a_0}\int_{t_1}^{t_0} \frac{c\,dt}{a(t)} = \frac{1}{a_0}\int_{t_1+\delta t_1}^{t_0+\delta t_0} = \frac{1}{a_0}\left[\int_{t_1+\delta t_1}^{t_1} + \int_{t_1}^{t_0} + \int_{t_0}^{t_0+\delta t_0}\right]$$

$$\Rightarrow \int_{t_1}^{t_1+\delta t_1} \frac{c\,dt}{a} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{a}$$

$$\Rightarrow \frac{\delta t_1}{a_1} = \frac{\delta t_0}{a_0} \Rightarrow \boxed{\frac{\delta t_0}{\delta t_1} = \frac{a_0}{a_1} = 1+z}$$

$$\Rightarrow \delta z = \left(\frac{a_0}{a_1}\right)\left[\frac{1+H_0\,\delta t_0}{1+H_1\,\delta t_1} - 1\right]$$

$$= (1+z)\,\delta t_0\left[H_0 - H_1/1+z\right]\left[1 - H_1\,\delta t_0/1+z\right]$$

$$= (1+z)\,\delta t_0\left[H_0 - H_1/1+z\right]$$

Problem 2 (Cont'd)

d) Friedmann equation

$$\left(\frac{\dot a}{a}\right) - \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

Continuity equation

$$\dot\rho + 3H(1+w)\rho = 0 \iff \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Then,

$$H^2 = H_0^2 \underset{1}{\underset{\shortparallel}{\Omega_0}} (1+z)^{3(1+w)}$$

or $$\boxed{H = H_0 (1+z)^{\frac{3}{2}(1+w)}}$$

This gives

$$\boxed{\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{\frac{3}{2}(1+w)}}$$

e) $\frac{dz}{dt} > 0 \iff \ln(1+z) > \frac{3}{2}(1+w)\ln(1+z)$

$\iff w < -\frac{1}{3}$

f) $w = 0$ $\quad \delta z = H_0\left[(1+z) - (1+z)^{3/2}\right]\delta t_0 = 10^{-5}$

$$\delta t_0 = \frac{10^{-5}}{(2^{3/2} - 2)H_0} = 177\,515 \text{ yr}$$

$w = -1$ $\quad \delta z = H_0\left[(1+z) - (1+z)^0\right]\delta t_0 = 10^{-5}$

$$\delta t_0 = \frac{10^{-5}}{(2-1)H_0} = 147\,000 \text{ yr}$$

Problem 3

(a) Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(T)$$

$T >> m_\chi$ , $\chi$'s are relativistic; $\gamma$'s and $\chi$'s contribute

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\left[\rho_\gamma(T) + \rho_\chi(T)\right]$$

$$= \frac{8\pi G}{3}\left[\frac{\pi^2}{30} g_\gamma T^4 + \frac{7}{8}\frac{\pi^2}{30} g_\chi T^4\right]$$

$$\boxed{H = \sqrt{\frac{8\pi G}{3}}\left[\frac{\pi^2}{30}\right]^{1/2}\left[g_\gamma + \frac{7}{8} g_\chi\right]^{1/2} T^2}$$

$T << m_\chi$ $\chi$'s drop from the sum above

(b) $\Gamma_\chi = n_\chi \langle \sigma v \rangle$ , $\left(n_\chi = \frac{3\zeta(3)}{4\pi^2} g_\chi T^3\right)$

$[L^{-3}]$ $[L^2]$ $[L T^{-1}]$

(c) $\frac{\Gamma_\chi}{H} > 1$

$$\Rightarrow \boxed{\langle \sigma v \rangle > \frac{4\pi^2}{3\zeta(3)}\sqrt{\frac{8\pi G}{3}}\sqrt{\frac{\pi^2}{30}}\frac{\left[g_\gamma + \frac{7}{8}g_\chi\right]^{1/2}}{g_\chi} \times \frac{1}{T}}$$

Problem 3 (Cont'd)

(d) $T < m_X$ $\quad n_X = g_X \left(\frac{m_X T}{2\pi}\right)^{3/2} e^{-m_X/T} \ll n_\gamma$

$\Rightarrow \Gamma_X \searrow$

$\Gamma(T_{dec}) = H(T_{dec})$

$\rightsquigarrow n_X(T_{dec}) \langle \sigma v \rangle = K T_{dec}^2 = K \beta^2 m_X^2$

$\rightsquigarrow \boxed{n_X = \frac{K\beta^2 m_X^2}{\langle \sigma v \rangle}}$

(e) $n_X / n_\gamma$ at decoupling

$n_\gamma = \frac{3\zeta(3)}{4\pi^2} g_\gamma T^3 = \alpha T^3$

$\frac{n_X}{n_\gamma} = \left(\frac{K}{\alpha}\right) \frac{\beta^2 m_X^2}{\langle \sigma v \rangle} \frac{1}{\beta^3 m_X^3} = \frac{K}{\alpha} \frac{1}{\beta \langle \sigma v \rangle m_X}$

(f) Present value of $n_X / n_\gamma$

$n_X \propto a^{-3}$ $\quad n_\gamma \propto a^{-3}$ $\quad$ Constant ratio!

(g) Present value of $\rho_X$

$\rho_{X_0} = n_{X_0} m_X = n_X^{dec} \times \left(\frac{n_{X_0}}{n_X^{dec}}\right) m_X = n_X^{dec} \left(\frac{a_{dec}}{a_0}\right)^3 m_X$

Problem 3 (Cont'd)

(g) So,

$$\rho_X(0) = n_X(\mathrm{dec}) \left( \frac{a_{dec}}{a_0} \right)^3 m_X$$

$$= n_\gamma(\mathrm{dec}) \left( \frac{a_{dec}}{a_0} \right)^3 \frac{K}{\alpha} \frac{1}{\beta \langle \sigma v \rangle m_X}$$

(h) $T \propto \frac{1}{a}$

$$\left.\begin{aligned} \rho_X &= \frac{1}{a^3} \\ \rho_\gamma &= \frac{1}{a^4} \end{aligned}\right\} \quad \frac{\rho_X}{\rho_\gamma} \propto a \, ; \quad \frac{\rho_X}{\rho_\gamma} = \left( \frac{\rho_X}{\rho_\gamma} \right)_{dec} \left( \frac{a}{a_{dec}} \right)$$

$$= \left( \frac{T_{dec}}{T} \right) \left( \frac{\rho_X}{\rho_\gamma} \right)_{dec}$$

$$\boxed{T_{eq} = T_{dec} \left( \frac{\rho_X}{\rho_\gamma} \right)_{dec}}$$

(i)