Master NPAC – Cosmology

February 5, 2019

Note 10% precision on numerical results is sufficient.

Useful quantities and formulae

- Present expansion rate: $H_0 = 100h\ \mathrm{km\,s^{-1}Mpc^{-1}}$ $h = 0.70 \pm 0.03$
- $\frac{c}{H_0} = 2998h^{-1}\mathrm{Mpc} = 3\ 10^{22}\mathrm{m}$
- $\frac{1}{H_0} = 1.0 \times 10^{10}h^{-1}\mathrm{yr} = 3 \times 10^{17}h^{-1}\mathrm{sec}$
- Present photon temperature: $kT_\gamma = 2.3 \times 10^{-4}\mathrm{eV}$
- Present photon energy density: $\rho_\gamma = (\pi^2/15)T_\gamma^4 = 0.26 \times 10^6\ \mathrm{eVm^{-3}}$
- Present photon number density: $n_\gamma = (2.4/\pi^2)T_\gamma^3 = 4.09 \times 10^8\mathrm{m^{-3}}$
- Present critical density: $\rho_{c0} = 3H_0^2c^2/(8\pi G) = 1.0 \times 10^{10}h^2\ \mathrm{eV\,m^{-3}}$
- Friedmann eqn.: $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad = H_0^2\rho/\rho_{c0} - \frac{k}{a^2}$
- Friedman for ΛCDM, $z \ll 1000$: $H^2 \sim H_0^2[\Omega_\Lambda + \Omega_M(1+z)^3 + \Omega_k(1+z)^2]$
- conservation of energy for a fluid of equation of state $p = w\rho$: $\dot{\rho} + 3H(1+w)\rho = 0$
- System of equations governing the dynamics of a scalar field in a FRW geometry:

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{1}$$

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 \tag{2}$$

$$\dot{H} = -\frac{1}{2M_{\mathrm{Pl}}^2}\dot{\phi}^2 \tag{3}$$

  where $V_\phi = \frac{dV}{d\phi}$ and $M_{\mathrm{Pl}} = 1/\sqrt{8\pi G}$. For simplicity, **we work in units in which** $M_{\mathrm{Pl}} = 1$.

- The energy density ρ and pressure P of the scalar field are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{5}$$

- The slow roll parameters defined in lectures are (with $M_{\rm Pl} = 1$):

$$\epsilon_V \equiv \frac{1}{2}\left(\frac{V_\phi}{V}\right)^2 \qquad \eta_V \equiv \frac{V_{\phi\phi}}{V} \tag{6}$$

- A different, and often more convenient set of slow-roll parameters (called the "Hubble" slow-roll parameters) are defined by

$$\epsilon_0 \equiv H_*/H \tag{7}$$

$$\epsilon_{i+1} = -\frac{d\ln|\epsilon_i|}{dN} \qquad i \geq 0, \tag{8}$$

  where H_* is the Hubble parameter at a chosen time (and hence a *constant*). The slow-roll parameter ϵ_1 was also introduced in lectures.

- The *number of e-folds before the end of inflation* is defined by

$$N(t) = \ln\frac{a_{end}}{a(t)} \tag{9}$$

  where a_{end} is the scale factor at the end of inflation.

Problem 1
Consider a flat universe with $H(z) = H_0 = \text{Cte}$. A galaxy at redshift z is observed.

(a) what is the present distance to the galaxy ?

(b) what was the distance to the galaxy when the photons we are now receiving were emitted ?

(c) what was the flight time of these photons from the galaxy to us?

A supernova explodes in the galaxy and emits a total of N photons.

(d) for a flat universe, what is the area of the sphere centered on the supernova and intercepting our position ?

(e) what is the number of photons detected by an observer on the sphere equipped with a detector of area A ?

(f) would the observer detect the same number of photons, more photons or fewer photons if the distance was the same, but $\Omega_k = 1 - \Omega_{\text{tot}} > 0$? Explain your answer.

Problem 2: Redshift Drift
An observer measures the redshift of a source at t_0. In this problem, we want to estimate by how much the source redshift has varied when we re-observe the same source at a time $t_0 + \delta t_0$.

(a) write the expression of the redshift $z(t_0)$ as a function of the scale parameter a, the time of observation t_0 and the time of emission t_1. Same question for $z(t_0 + \delta t_0)$.

(b) Show that, at first order in δt, one can express the redshift variation $\delta z = z(t_0 + \delta t_0) - z(t_o)$ as a function of $H(t_0)$, $\mathrm{H(t_1)}$, t_0, t_1 and a.

(c) The comoving coordinate of the source is constant (we neglect its peculiar velocity). There is therefore a simple relation between the time intervals δt_0 and δt_1. Write this relation, and show that

$$\frac{dz}{dt_0} = H_0 \times (1 + z) - H(z) \tag{10}$$

(d) Let's assume a flat, single component, Universe. The equation of state of this component is $p = w\rho$. Derive from the Friedmann equation the evolution of H as a function of z. Show that one can express dz/dt_0 as a function of H_0, z and w.

(e) Describe the evolution of z as a function of time. Do we always have $dz/dt_0 > 0$? If not, for which range of w do we observe an increase (resp. decrease) of z ?

(f) Assume you live in a flat matter-dominated Universe, with $H_0 = 68\ \text{kms}^{-1}\text{Mpc}^{-1}$. You observe a galaxy at $z = 1$. How long do have to wait until you can detect a relative redshift variation of 10^{-5} ?

(g) Same question for a flat, vacuum-energy-dominated Universe.

Problem 3: Weakly Interacting Massive Particles

In this question you can, if you like, ignore uninteresting numerical factors ($2\pi = 1$) and set $\hbar = c = 1$ and $G = 1/m_{planck}^2$.

Consider a simple universe consisting of photons of number density n_γ and a massive spin 1/2 fermion, χ, of mass m_χ and number density n_χ . Besides elastic scattering and bremstrahlung, the only permitted reactions are $\chi\chi \leftrightarrow \gamma\gamma$ (χ's and photons are their own antiparticles). The number densities, n_χ and n_γ, will take on thermal values at the temperature, T if the forward and backwards rates for $\chi\chi \leftrightarrow \gamma\gamma$ are greater than the expansion rate, $H(T)$.

(a) Write an expression for the expansion rate as a function of the temperature for $T \gg m_\chi$ and for $T \ll m_\chi$, and assuming that photons and χ are in thermal equilibrium. Assume that only relativistic species contribute to the energy density.

(b) Write an expression for the annhilation rate, Γ_χ, of χ, i.e. the reciprocal of the mean time before a χ finds another χ and annihilates. The expression should depend on the number density and on the annihilation cross-section times velocity, σv (assumed velocity- and temperature independent as is often the case for exothermal reactions).

(c) Consider a temperature $T > m_\chi$. Suppose that the χ are in thermal equilibrium so that $n_\chi \sim T^3$. How large must σv be for the annhilation rate to be greater than the expansion rate.

(d) Suppose that the conditions from (c) are satisfied. As the temperature drops below m_χ the number density of χ drops below that of the photons which now dominate the energy density. As n_χ falls, the annhilation rate falls. The decoupling temperature, T_{dec} is defined as the temperature where $\Gamma_\chi(T_{dec}) = H(T_{dec})$ and we write $T_{dec} = \beta m_\chi$, where $\beta < 1$ is a numerical factor that we will estimate shortly. By setting the annihilation rate equal to the expansion rate, estimate the number density $n_\chi(T_{dec})$ as a function of σv, m_χ, and β. (Hint: you do not need an explicit form for $n_\chi(T)$ to do this.)

(e) What is the χ-photon ratio, n_χ/n_γ at decoupling as a function of σv, m_χ and β?

(f) Assuming that there are no annihilation after T_{dec}, what is the present value of n_χ/n_γ.

(g) What is the present value of ρ_χ

(h) At what temperature does the universe become matter dominated?

(i) (Just to test your mathematical dexterity). For $T < m_\chi$, the equilibrium number of χ is given by $n_\chi = 2(Tm_\chi/2\pi)^{3/2} exp(-m_\chi/T)$. Find an expression for β that depends on σv and on $log\beta$. For a given σv, this expression can be solved iteratively for β.

Problem 4:

i) Assume that the scale factor $a(t)$ of the universe is given by a power law, $a \propto t^q$. For which values of q does inflation occur? More generally, show that inflation occurs when $-\dot{H} < H^2$ or equivalently when the comoving Hubble radious $(aH)^{-1}$ decreases in time.

$$\frac{d}{dt}\frac{1}{aH} < 0 \,.$$

ii) The slow-roll parameter $\epsilon_1 \equiv -\frac{\dot{H}}{H^2}$. Show that it can be rewritten as $\epsilon_1 = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}$. Hence, using the Friedmann equation, deduce that

$$H^2\left(1 - \frac{\epsilon_1}{3}\right) = \frac{V}{3}\,, \qquad \frac{\ddot{a}}{a} = H^2(1 - \epsilon_1).$$

For what values of ϵ_1 does inflation occur?

iii) Show that conformal time η is given by

$$\eta = \int \frac{da}{a^2 H} = -\frac{1}{aH} + \int \frac{\epsilon_1}{a^2 H} da$$

(hint: integrate by parts). Assuming ϵ_1 is constant, which is equivalent to working to first order in slow-roll parameters, deduce that $a \simeq -\frac{1}{\eta H(1-\epsilon_1)}$.

iv) When considering perturbations of the field $\phi(t, \vec{x}) = \bar{\phi}(t) + \chi(t, \vec{x})$ around the homogeneous solution $\bar{\phi}(t)$, and neglecting the backreaction of these fields on the FRLW geometry, a similar calculation to that done in lectures yields the following power-spectrum $\mathcal{P}_\chi(k)$ for super-Horizon modes:

$$\mathcal{P}_\chi(k) \;\propto\; \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu}$$

where

$$\nu^2 = \frac{9}{4} + 3\epsilon_1$$

1. Comment on the limit $\epsilon_1 = 0$.
2. Deduce the spectral index defined by

$$n_s - 1 = \frac{d \ln \mathcal{P}_\chi(k)}{d \ln k}$$

in terms of the slow-roll parameter ϵ_1.

3. Now express n_s in terms N for the potential $V = \alpha\phi$.
4. Calculate n_s for $N = 60$.