Mid term exam in QFT

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We consider a 4-vector field $A_\mu(x)$. We want first to build the simplest lagrangian $\mathcal{L}_A$ for $A_\mu(x)$.
1. $\mathcal{L}_A$ depends of course on $A_\mu(x)$. On which other quantities does it depend ? How many quantities does this represent in total ?

2. We first consider the kinetic part of $\mathcal{L}_A$, that is, the part that involves derivatives of the field. Show that there is no acceptable term for $\mathcal{L}_A$ involving either only one field and one derivative or one field and two derivatives or two fields and one derivative.

3. Find the term(s) with two fields and two derivatives that can contribute to $\mathcal{L}_A$.

4. We are now interested in the simplest (that is, of lowest degree) terms contributing to $\mathcal{L}_A$ and that do not involve derivatives of the field. Show that the first acceptable term is of degree two in the field. We call $-m^2/2$ the coefficient in front of this term in $\mathcal{L}_A$.

5. The simplest lagrangian $\mathcal{L}_A$ is a linear combination of the terms found above. We now impose that this lagrangian be invariant under the gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{1}$$

with $\alpha(x)$ any function. Show that the resulting lagrangian $\mathcal{L}_A$ can be rewritten in terms of

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

For future convenience, we normalize the field so that the coefficient in front of the term thus obtained is $-1/4$. We call $F_{\mu\nu}$ the electomagnetic tensor and $\mathcal{L}_A$ the lagrangian of pure electrodynamics (no coupling to a source).

6. Find the equations of motion (Euler-Lagrange) of the theory. Are they gauge invariant ?

7.a How many components does $F_{\mu\nu}$ involve ? We define as usual

$$A^\mu = (A^0 = \phi, \vec{A}) \tag{3}$$

with ϕ the scalar potential and $\vec{A}$ the vector potential with coordinates A^i, $i = 1, 2, 3$. The electric and magnetic fields are defined by

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla} \wedge \vec{A} \tag{4}$$

with by definition (be careful to this definition) :

$$\vec{\bigtriangledown} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}). \tag{5}$$

We choose to define the coordinates of all 3-vectors $\vec{A}$, $\vec{E}$ and $\vec{B}$ as contravariant coordinates. For instance the component number i of $\vec{A}$ is $(\vec{A})_i = A^i$.

7.b Compute $F^{\mu\nu}$ in terms of the components of the electric and magnetic field. Write the matrix $F^{\mu\nu}$ in terms of E^i and B^i.

7.c Find the gauge transformations of $\phi(x)$ and $\vec{A}(x)$ and the equations of motion in terms of $\vec{E}$ and $\vec{B}$.

7.d [This is a supplementary question : do not work on it until you have answered to all the questions below]

Compute $\mathcal{L}_A$ and the corresponding hamiltonian in terms of $\vec{E}$ and $\vec{B}$. Which property can you see on this hamiltonian which will be important to maintain in the quantum case (with, maybe, some minor changes).

8. We now consider a Dirac field $\psi(x)$.

8.a Show that the lagrangian $\mathcal{L}_{0,\psi}$ for a free spin 1/2 particle of mass m associated with the field ψ :

$$\mathcal{L}_{0,\psi} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \tag{6}$$

is invariant under the transformation :

$$\psi(x) \to \psi'(x) = e^{-iq\alpha}\psi(x) \tag{7}$$

where q is a fixed real parameter and α a number that can take any real values.

8.b Compute the Noether charge associated with this $U(1)$ symmetry.

8.c Is $\mathcal{L}_{0,\psi}$ invariant under

$$\psi(x) \to \psi'(x) = e^{-iq\alpha(x)}\psi(x) \tag{8}$$

where $\alpha(x)$ is now a function of x ?

9. We now consider the lagrangian $\mathcal{L}_0 = \mathcal{L}_A + \mathcal{L}_{0,\psi}$. In order to couple ψ and A_μ, we want add to $\mathcal{L}_0$ an intercation term $\mathcal{L}_{\text{int}}$ between these fields. We shall choose for $\mathcal{L}_{\text{int}}$ the simplest monomial(s) that are functions of both ψ and A_μ, that is, the monomial(s) of lowest degree in these fields. The final lagrangian will be called $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$. Find the simplest (in the sense above) interaction term(s) between ψ and A_μ.

10. To fully determine $\mathcal{L}$, we constrain it to be gauge invariant, that is, to be invariant under the combined transformations of Eqs. (1) and (8).

10.a Show that by adjusting the coupling constant(s) in front of $\mathcal{L}_{\text{int}}$, it is indeed possible to make $\mathcal{L}$ gauge invariant.

10.b We define $D_\mu = \partial_\mu + ia_\mu$. Determine a_μ such that the lagrangian $\mathcal{L}_\psi = \mathcal{L}_{0,\psi} + \mathcal{L}_{\text{int}}$ can be obtained from $\mathcal{L}_{0,\psi}$ alone by the substitution $\partial_\mu \to D_\mu$:

$$\mathcal{L}_\psi = \mathcal{L}_{0,\psi}{}_{|_{\partial_\mu\psi \to D_\mu\psi}} = \mathcal{L}_{0,\psi} + \mathcal{L}_{\text{int}}. \tag{9}$$

10.c Show that $D_\mu \psi$ transforms under the gauge transformation (1) and (8) as ψ which is the very reason why $\mathcal{L}_{0,\psi} + \mathcal{L}_{\rm int}$ is gauge invariant. We call D_μ the covariant derivative (it co-varies, that is, it varies as ψ).

10.d Compute $[D_\mu, D_\nu]$ and conclude that it is gauge invariant.

11. Maxwell equations.

11.a Compute the equations of motion of A_μ obtained with $\mathcal{L}$.

11.b Check that they are consistent with the conservation of the Noether charge. What do the Noether current and charge represent physically ?

11.c [This is a supplementary question : do not work on it until you have answered to all the questions below]

Rewrite the equations of motion in terms of the electric and magnetic fields.

12. We now want to generalize the U(1) gauge symmetry found above to other gauge groups. We thus consider a theory with N Dirac bispinors $\psi_1, \psi_2, \cdots \psi_N$ of mass $m_1, m_2, \cdots m_N$. We define

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{pmatrix} \tag{10}$$

and $\bar\psi(x) = (\bar\psi_1(x), \bar\psi_2(x), \cdots \bar\psi_N(x))$. The free lagrangian of this model is

$$\mathcal{L}_{0,\psi} = \mathcal{L}_{0,\psi_1} + \cdots + \mathcal{L}_{0,\psi_N} . \tag{11}$$

Write $\mathcal{L}_{0,\psi}$ in terms of ψ and in a matrix form.

Remark : It is important in the following to understand the matrix structure of the equations we manipulate : a $p \times p$ matrix can have matrices as matrix elements. This is similar to the 4×4 Dirac matrices γ^i that can be written (in Weyl's representation) as 2×2 matrices with Pauli matrices as matrix elements :

$$\gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix} . \tag{12}$$

13. In the following, we shall admit Schur's lemma that states that if a matrix commutes with all the matrices of a group G, then this matrix is proportional to the identity.

We now transform $\psi_i(x)$ according to :

$$\psi_i(x) \to \psi'_i(x) = U_{ij} \psi_j(x) \tag{13}$$

where U is a $N \times N$ unitary matrix belonging to SU(N).[1]

1. Notice that (i) in the $N = 2$ case, this SU(N =2) has nothing to do with the SU(2) associated with the rotations in the three dimensional spatial part of the minkowskian space-time : it is an "internal" transformations acting on the fields and that does not affect the coordinates x^μ ; (ii) the matrix elements of U are themselves matrices since Eq.(13) really means : $(\psi_i)_\alpha = U_{ij}\,(\psi_j)_\alpha = U_{ij}\delta_{\alpha\beta}\,(\psi_j)_\beta$ with $i, j = 1, \cdots, N$ and $\alpha, \beta = 1, \cdots, 4$.

Under which condition on $m_1, \cdots, m_N$ is $\mathcal{L}_{0,\psi}$ invariant under the transformations (13) for all $U \in \mathrm{SU}(N)$?

14. We assume that the condition derived on the masses above holds true and we now want to gauge the previous transformation (13), that is, to make U a function of x :

$$\psi(x) \to \psi'(x) = U(x)\psi(x). \tag{14}$$

14.a Is $\mathcal{L}_{0,\psi}$ invariant under these gauge transformations ?

14.b We try the same trick as in question 10.b that consists in replacing in $\mathcal{L}_{0,\psi}$ the ordinary derivative ∂_μ by D_μ where $D_\mu = \partial_\mu + iA_\mu(x)$ and A_μ a 4-vector field. The idea is of course to make the resulting lagrangian $\mathcal{L}_\psi$ defined by

$$\mathcal{L}_\psi = \mathcal{L}_{0,\psi}{}_{|\partial_\mu\psi \to D_\mu\psi} \tag{15}$$

become gauge invariant.

Can we find a suitable transformation of A_μ such that $\mathcal{L}_\psi$ is invariant under the gauge transformation (14) ?

15. The problem encountered above to make $\mathcal{L}_\psi$ gauge invariant should suggest you to try a different replacement $\partial_\mu \to D_\mu$ where $D_\mu = \partial_\mu + i\mathcal{A}_\mu$ with $\mathcal{A}_\mu$ being a matrix whose matrix elements are 4-vectors.

15.a What can be the dimension of the matrix $\mathcal{A}_\mu$?

15.b Find the transformation law of $\mathcal{A}_\mu$ such that $D_\mu = \partial_\mu + i\mathcal{A}_\mu$ is a covariant derivative, that is, $D'_\mu\psi' = U(x)D_\mu\psi$.

16. From now on, we take $N = 2$. We recall that

- for an SU(2) matrix $U(\vec{\theta})$ and its associated SO(3) matrix $R(\vec{\theta})$:

$$U^{-1}\sigma_i U = R_{ij}\sigma_j; \tag{16}$$

- the set of matrices $\sigma_a = (I_2, \sigma_1, \sigma_2, \sigma_3)$ where I_2 is the identity matrix and the σ_i's are the Pauli matrices, is a basis of the four dimensional complex space of general 2×2 matrices ;

- $\sigma_i\sigma_j = \delta_{ij}I_2 + i\epsilon_{ijk}\sigma_k$.

16.a Consider an infinitesimal transformation matrix $U(x) \in \mathrm{SU}(2)$. Show that $(\partial_\mu U)U^{-1}$ belongs to the Lie algebra of SU(2), that is, is a linear combination of the generators of the group.

16.b [You can skip this question if you don't have enough time, admit the result and go directly to 16.c]

Show that for any matrix $U(x) \in \mathrm{SU}(2)$, $(\partial_\mu U)U^{-1}$ belongs to the Lie algebra of SU(2).

16.c From the questions above, conclude that it is necessary to introduce only three real (and not four complex) independent 4-vector fields in the matrix $\mathcal{A}_\mu$ and that $\mathcal{A}_\mu$ is in the Lie algebra of SU(2).

16.d By considering an infinitesimal transformation, show that the transformation of $\mathcal{A}_\mu$ depends only on the Lie algebra of the group SU(2) and not on the fact that ψ transforms under a particular representation of SU(2) (in the present case, SU(2) itself). Why is this important if we want to generalize the model to other sets of Dirac fields spanning other representations of SU(2) ?

17. We now want to compute the lagrangian $\mathcal{L}_A$ which is also gauge invariant.

17.a From the transformation of $D_\mu \psi$ in terms of U, find the transformation of D_μ itself.

17.b Compute $F_{\mu\nu}$ defined by $iF_{\mu\nu} = [D_\mu, D_\nu]$. Can we decompose $F_{\mu\nu}$ on the generators of SU(2) ?

17.c Find the transformation law of $F_{\mu\nu}$ under an SU(2) gauge transformation. Is it gauge invariant as in the U(1) case ?

17.d Find a gauge and Lorentz invariant term which is quadratic in $F_{\mu\nu}$. This is, up to a constant, the kinetic term for $\mathcal{A}_\mu$.