1.8 Traces of gamma matrices

Now compute the traces using the fundamental relation $\left\{\gamma^{\alpha},\gamma^{\beta}\right\}=2\eta^{\alpha\beta}$, and the cyclic property of the trace, $tr(A\dots BC)=tr(CA\dots B)$. First, we can show that the trace of the product of any odd number of γ-matrices vanishes by using $\gamma_5^2=1$ and $\{\gamma_5,\gamma^a\}=0$,

$$\begin{aligned}
tr\left(\underbrace{\gamma^{\alpha}\dots\gamma^{\beta}}_{2n+1}\right) &= tr\left(1\gamma^{\alpha}\dots\gamma^{\beta}\right)\\
&= tr\left(\gamma_5\gamma_5\gamma^{\alpha}\dots\gamma^{\beta}\right)\\
&= -tr\left(\gamma_5\gamma^{\alpha}\gamma_5\dots\gamma^{\beta}\right)\\
&= (-1)^{2n+1}tr\left(\gamma_5\gamma^{\alpha}\dots\gamma^{\beta}\gamma_5\right)\\
&= (-1)^{2n+1}tr\left(\gamma_5\gamma_5\gamma^{\alpha}\dots\gamma^{\beta}\right)\\
&= -tr\left(\gamma^{\alpha}\dots\gamma^{\beta}\right)\\
&= 0
\end{aligned}$$

For even products, we will need traces of products of $2,4,6$ and 8 gamma matrices.

$$\begin{aligned}
tr\left(\gamma^{\alpha}\gamma^{\beta}\right) &= tr\left(-\gamma^{\beta}\gamma^{\alpha}+2\eta^{\alpha\beta}1\right)\\
&= -tr\left(\gamma^{\beta}\gamma^{\alpha}\right)+2\eta^{\alpha\beta}tr(1)\\
&= -tr\left(\gamma^{\alpha}\gamma^{\beta}\right)+8\eta^{\alpha\beta}\\
tr\left(\gamma^{\alpha}\gamma^{\beta}\right) &= 4\eta^{\alpha\beta}
\end{aligned}$$

and

$$\begin{aligned}
tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right) &= tr\left(\left(-\gamma^{\beta}\gamma^{\alpha}+2\eta^{\alpha\beta}1\right)\gamma^{\mu}\gamma^{\nu}\right)\\
&= -tr\left(\gamma^{\beta}\gamma^{\alpha}\gamma^{\mu}\gamma^{\nu}\right)+2\eta^{\alpha\beta}tr(\gamma^{\mu}\gamma^{\nu})\\
&= -tr\left(\gamma^{\beta}\left(-\gamma^{\mu}\gamma^{\alpha}+2\eta^{\mu\alpha}\right)\gamma^{\nu}\right)+2\eta^{\alpha\beta}tr(\gamma^{\mu}\gamma^{\nu})\\
&= tr\left(\gamma^{\beta}\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}\right)-2\eta^{\mu\alpha}tr\left(\gamma^{\beta}\gamma^{\nu}\right)+2\eta^{\alpha\beta}tr(\gamma^{\mu}\gamma^{\nu})\\
&= tr\left(\gamma^{\beta}\gamma^{\mu}\left(-\gamma^{\nu}\gamma^{\alpha}\right)+2\eta^{\nu\alpha}\right)-2\eta^{\mu\alpha}tr\left(\gamma^{\beta}\gamma^{\nu}\right)+2\eta^{\alpha\beta}tr(\gamma^{\mu}\gamma^{\nu})\\
&= -tr\left(\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\right)+2\eta^{\nu\alpha}tr\left(\gamma^{\beta}\gamma^{\mu}\right)-2\eta^{\mu\alpha}tr\left(\gamma^{\beta}\gamma^{\nu}\right)+2\eta^{\alpha\beta}tr(\gamma^{\mu}\gamma^{\nu})\\
2tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right) &= 2\eta^{\nu\alpha}tr\left(\gamma^{\beta}\gamma^{\mu}\right)-2\eta^{\mu\alpha}tr\left(\gamma^{\beta}\gamma^{\nu}\right)+2\eta^{\alpha\beta}tr(\gamma^{\mu}\gamma^{\nu})\\
tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right) &= 4\eta^{\nu\alpha}\eta^{\beta\mu}-4\eta^{\mu\alpha}\eta^{\beta\nu}+4\eta^{\alpha\beta}\eta^{\mu\nu}
\end{aligned}$$

For six, we use the simple pattern to more quickly find

$$\begin{aligned}
tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= tr\left(\left(-\gamma^{\beta}\gamma^{\alpha}+2\eta^{\alpha\beta}1\right)\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right)\\
&= tr\left(-\gamma^{\beta}\left(2\eta^{\alpha\mu}-\gamma^{\mu}\gamma^{\alpha}\right)\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}+2\eta^{\alpha\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right)\\
&= tr\left(-\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\alpha}+2\eta^{\alpha\sigma}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-2\eta^{\alpha\rho}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}+2\eta^{\alpha\nu}\gamma^{\beta}\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}-2\eta^{\alpha\mu}\gamma^{\beta}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}+2\eta^{\alpha\beta}\gamma^{\mu}\gamma
\end{aligned}$$

and from here we can use the result for the trace of four,

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= tr\left(\eta^{\alpha\sigma}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho - \eta^{\alpha\rho}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma + \eta^{\alpha\nu}\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\sigma - \eta^{\alpha\mu}\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma + \eta^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= 4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\rho\beta}\eta^{\mu\nu}\right) - 4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma} - \eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\beta\sigma}\eta^{\mu\rho}\right) - 4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\nu\sigma} + \eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)
\end{aligned}
$$

or, perhaps more mnemonically,

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - 4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\nu\sigma} + \eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\beta\sigma}\eta^{\mu\rho}\right) - 4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma} - \eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\rho\beta}\eta^{\mu\nu}\right)
\end{aligned}
$$

From this, if we don't run out of Greek letters, we can immediately write the result for eight gammas:

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\tau\right) &= 4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - 4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\nu\sigma} + \eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\beta\sigma}\eta^{\mu\rho}\right) - 4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma} - \eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\rho\beta}\eta^{\mu\nu}\right)
\end{aligned}
$$