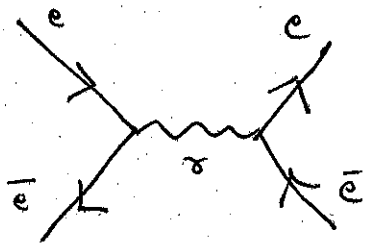


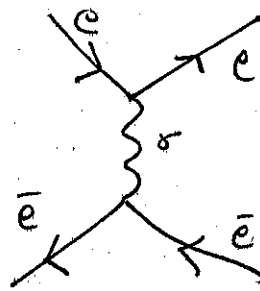
2. QED cross-sections

2.1 $e^+e^- \rightarrow e^+e^-$ (Bhabha, 1936)

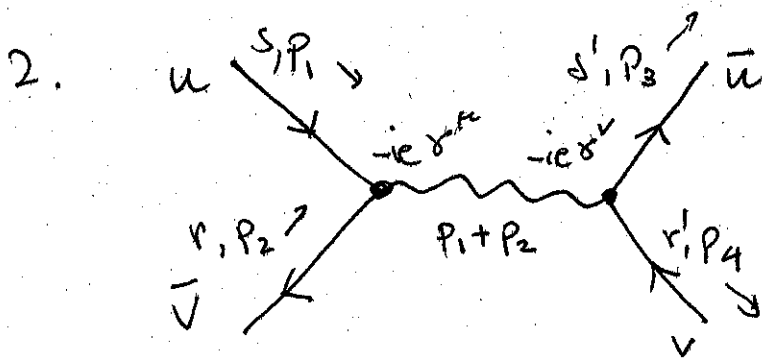
1. At tree-level we have the two diagrams:



s-channel

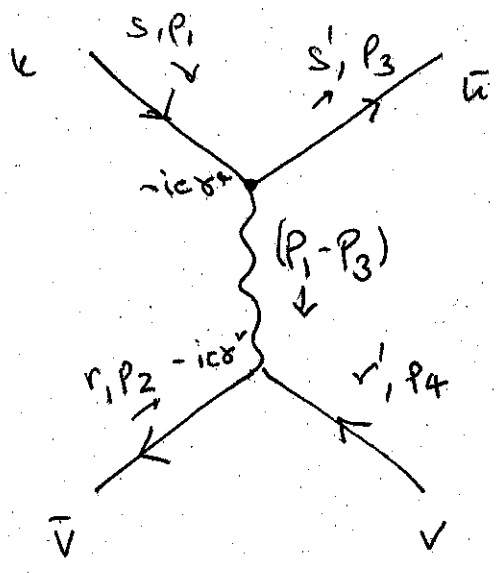


t-channel



s-channel
Amplitude

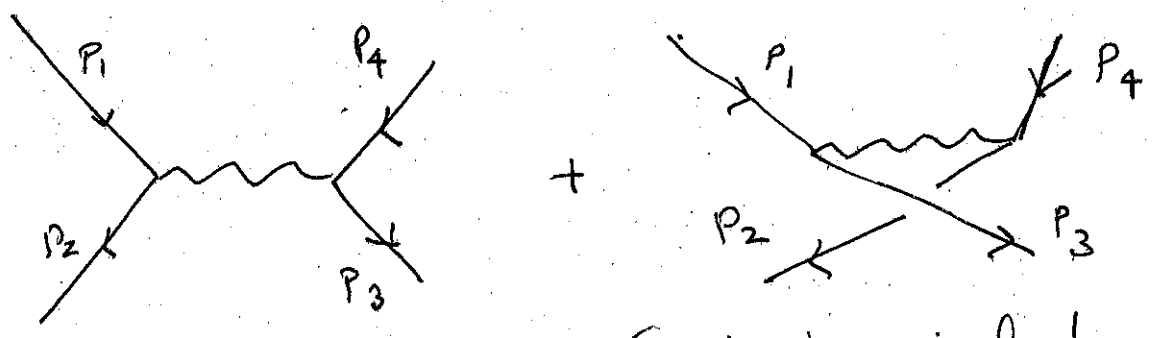
$$A_s = (-i) \bar{v}(p_2)^r (-ie\gamma^\mu) u^s(p_1) \left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2} \right) \times \bar{u}^{s'}(p_3) (-ie\gamma^\nu) v^{r'}(p_4)$$



t-channel Amplitude

$$A_t = (-i) \bar{v}(p_2) (-i\epsilon\sigma^\nu) v(p_4) \left(-\frac{i g_{\mu\nu}}{(p_1 - p_3)^2} \right) \bar{u}(p_3) (-i\epsilon\sigma^\mu) u(p_1)$$

The two terms have a relative - sign due to Fermi statistics: in fact, we can show the two diagrams in the following way:



(this is equivalent to the t-channel diagram, since p_1 is connected to p_3 and p_2 to p_4)

clearly they are related by exchange of one fermionic leg \Rightarrow - sign!

$$A_s^+ A_t = -\frac{e^4}{4St} \left\{ \begin{aligned} &(\bar{u}(p_1) \gamma^\mu v(p_2)) (\bar{v}(p_4) \gamma_\mu u(p_3)) \\ &(\bar{v}(p_2) \gamma^\nu v(p_4)) (\bar{u}(p_3) \gamma_\nu u(p_1)) \end{aligned} \right.$$

$$\rightarrow -\frac{e^4}{4St} \left[\begin{aligned} &\bar{u}_\alpha^s(p_1) \gamma_{\alpha\beta}^\mu v_\beta^r(p_2) \bar{v}_\lambda^r(p_2) \gamma_{\lambda\delta}^\nu v_\delta^{r'}(p_4) \bar{v}_\epsilon^{r'}(p_4) \gamma_{\mu\epsilon\sigma} u_\sigma^s(p_3) \\ &\bar{u}_\xi^s(p_3) \gamma_{\nu\xi\eta} u_\eta^s(p_1) \end{aligned} \right]$$

$$= -\frac{e^4}{4St} \left[\begin{aligned} &(\not{p}_1 + m)_{\eta\alpha} \gamma_{\alpha\beta}^\mu (\not{p}_2 - m)_{\beta\lambda} \gamma_{\lambda\delta}^\nu (\not{p}_4 - m)_{\delta\epsilon} \\ &\gamma_{\mu\epsilon\sigma} (\not{p}_3 + m)_{\sigma\xi} \gamma_{\nu\xi\eta} \end{aligned} \right]$$

$$= -\frac{e^4}{4St} \text{Tr} \left[\begin{aligned} &(\not{p}_1 + m) \gamma^\mu (\not{p}_2 - m) \gamma^\nu (\not{p}_4 - m) \gamma_\mu \\ &(\not{p}_3 + m) \gamma_\nu \end{aligned} \right]$$

so now it is a single trace

now:

$$\gamma^\mu (\not{p}_2 - m) \gamma^\nu = -\gamma^\mu \gamma^\nu (\not{p}_2 - m) + \gamma^\mu \{ \not{p}_2 - m, \gamma^\nu \}$$

$$A = A_t + A_s = (-i)(-ie)^2(-i) \times$$

$$\left[\frac{\bar{V}^r(p_2) \gamma^\mu U^s(p_1) \bar{U}^{s'}(p_3) \gamma_\mu V^{r'}(p_4)}{(p_1 + p_2)^2} - \frac{\bar{V}^r(p_2) \gamma^\mu V^{r'}(p_4) \bar{U}^{s'}(p_3) \gamma_\mu U^s(p_1)}{(p_1 - p_3)^2} \right]$$

introduce $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$

$$A = e^2 \left[\frac{1}{s} \bar{V}^r(p_2) \gamma^\mu U^s(p_1) \bar{U}^{s'}(p_3) \gamma_\mu V^{r'}(p_4) - \frac{1}{t} \bar{V}^r(p_2) \gamma^\mu V^{r'}(p_4) \bar{U}^{s'}(p_3) \gamma_\mu U^s(p_1) \right]$$

3. We want to compute

$$\frac{1}{4} \sum_{\text{spins}} |A|^2 = \frac{1}{4} \sum_{\substack{s, s' \\ r, r'}} \left(|A_s|^2 + |A_t|^2 + A_s^\dagger A_t + A_t^\dagger A_s \right)$$

where: $A_s = \frac{e^2}{s} \bar{V}(p_2) \gamma^\mu U(p_1) \bar{U}(p_3) \gamma_\mu V(p_4)$

$A_t = -\frac{e^2}{t} \bar{V}(p_2) \gamma^\mu V(p_4) \bar{U}(p_3) \gamma_\mu U(p_1)$

Let us consider each of the four terms: 5

$$|A_s|^2 = \frac{e^4}{S^2} (\bar{v}^{r'}(p_4) \gamma_\mu u^{s'}(p_3)) (\bar{u}^{s'}(p_3) \gamma_\nu v^{r'}(p_4)) \\ (\bar{u}^s(p_1) \gamma^\mu v^r(p_2)) (\bar{v}^r(p_2) \gamma^\nu u^s(p_1))$$

Using the spin sum-rules:

$$\sum_s u_\alpha^s(p) \bar{u}_\beta^s(p) = \not{p}_{\alpha\beta} + m \delta_{\alpha\beta} \quad (\not{p}_{\alpha\beta} \equiv \gamma^\mu p_\mu)$$

$$\sum_s v_\alpha^s(p) \bar{v}_\beta^s(p) = \not{p}_{\alpha\beta} - m \delta_{\alpha\beta}$$

and ordering the 4 factors above in a cyclic fashion, we find: (recall $\not{p} \equiv \gamma^\mu p_\mu$)

$$\frac{1}{4} \sum_{\substack{s, s' \\ r, r'}} |A_s|^2 = \frac{e^4}{4S^2} \text{Tr} [(\not{p}_4 - m) \gamma_\mu (\not{p}_3 + m) \gamma_\nu] \\ \text{Tr} [(\not{p}_1 + m) \gamma^\mu (\not{p}_2 - m) \gamma^\nu]$$

Now we use the trace identities:

$$\text{Tr} (\text{odd-number of } \gamma\text{'s}) = 0$$

$$\text{Tr} (\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

$$\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4 (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

So :

$$\begin{aligned}
 \text{Tr}[(\not{P}_4 - m)\gamma_\mu(\not{P}_3 + m)\gamma_\nu] &= \text{Tr}(\not{P}_4\gamma_\mu\not{P}_3\gamma_\nu) - m^2\text{Tr}\gamma_\mu\gamma_\nu \\
 &= P_4^\rho P_3^\sigma \text{Tr}(\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu) - m^2\text{Tr}(\gamma_\mu\gamma_\nu) = \\
 &= 4 P_4^\rho P_3^\sigma [g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}] - 4m^2 g_{\mu\nu} \\
 &= 4(P_{4\mu}P_{3\nu} + P_{4\nu}P_{3\mu} - P_4 \cdot P_3 g_{\mu\nu} - m^2 g_{\mu\nu})
 \end{aligned}$$

Similarly :

$$\begin{aligned}
 \text{Tr}[(\not{P}_1 + m)\gamma^\mu(\not{P}_2 - m)\gamma^\nu] &= \\
 &= 4(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - P_1 \cdot P_2 g^{\mu\nu} - m^2 g^{\mu\nu})
 \end{aligned}$$

Contrasting the two terms we find:

$$\frac{1}{4} \sum_{\text{spins}} |A_s|^2 = \frac{e^4}{4s^2} \cdot 16 \left(P_{4\mu}P_{3\nu} + P_{4\nu}P_{3\mu} - (m^2 + P_3 \cdot P_4)g_{\mu\nu} \right) \left(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - (m^2 + P_1 \cdot P_2)g^{\mu\nu} \right)$$

call: $P_{ij} \equiv P_i \cdot P_j$

$$= \frac{4e^4}{s^2} \left\{ 2P_{14}P_{23} + 2P_{24}P_{13} - 2(m^2 + P_{12})P_{34} - 2(m^2 + P_{34})P_{12} + 4(m^2 + P_{12})(m^2 + P_{34}) \right\}$$

$$= \frac{4e^4}{s^2} \left\{ 2P_{14}P_{23} + 2P_{24}P_{13} + 2m^2 P_{34} + 2m^2 P_{12} + 4m^4 \right\}$$

Finally:

$$\frac{1}{4} \sum_{\text{spins}} |A_s|^2 = \frac{8e^4}{s^2} \left[P_{14} P_{23} + P_{13} P_{24} + m^2 P_{12} + m^2 P_{34} + 2m^4 \right]$$

(this is the same as in $e^+e^- \rightarrow \mu^+\mu^-$ except that here the masses are all equal).

Now consider the $|A_t|^2$ term:

$$|A_t|^2 = \frac{e^4}{t^2} \left(\bar{V}(p_4) \gamma^\mu V(p_2) \right) \left(\bar{V}(p_2) \gamma^\nu V(p_4) \right) \left(\bar{u}(p_1) \gamma_\mu u(p_3) \right) \left(\bar{u}(p_3) \gamma_\nu u(p_1) \right)$$

Summing over polarisations:

$$\frac{1}{4} \sum_{\text{spins}} |A_t|^2 = \frac{e^4}{4t^2} \text{Tr} \left[(\not{p}_2 - m) \gamma^\nu (\not{p}_4 - m) \gamma^\mu \right] \text{Tr} \left[(\not{p}_3 + m) \gamma_\nu (\not{p}_1 + m) \gamma_\mu \right]$$

$$= \frac{e^4}{4t^2} \left[\text{Tr} [\not{p}_2 \gamma^\nu \not{p}_4 \gamma^\mu] + m^2 \text{Tr} (\gamma^\mu \gamma^\nu) \right] \left[\text{Tr} [\not{p}_3 \gamma_\nu \not{p}_1 \gamma_\mu] + m^2 \text{Tr} (\gamma_\mu \gamma_\nu) \right]$$

$$= \frac{4e^4}{t^2} \left(P_{2\nu} P_{4\mu} + P_{2\mu} P_{4\nu} + (m^2 - P_2 P_4) g_{\mu\nu} \right) \left(P_{3\nu} P_{1\mu} + P_{3\mu} P_{1\nu} + (m^2 - P_1 P_3) g_{\mu\nu} \right) =$$

$$= \frac{8e^4}{t^2} (P_{23} P_{14} + P_{12} P_{34} - m^2 (P_{24} + P_{13}) + 2m^4)$$

Cross terms

take first : $\frac{1}{4} \sum_{\text{spins}} A_t^+ A_s$

$$A_t^+ A_s = -\frac{e^4}{St} \left(\bar{v}^{r'}(p_4) \gamma^\mu v^r(p_2) \right) \left(\bar{u}^s(p_1) \gamma_\mu u^{s'}(p_3) \right) \\ \left(\bar{u}^{s'}(p_3) \gamma^\nu v^{r'}(p_4) \right) \left(\bar{v}^r(p_2) \gamma_\nu u^s(p_1) \right)$$

reshuffle:

$$= -\frac{e^4}{St} \left[\bar{v}^r(p_2) \gamma_\nu u^s(p_1) \bar{u}^{s'}(p_1) \gamma_\mu u^{s'}(p_3) \bar{u}^{s'}(p_3) \gamma^\nu v^{r'}(p_4) \right. \\ \left. \bar{v}^{r'}(p_4) \gamma^\mu v^r(p_2) \right]$$

Now summing over r, s, r', s' gives a single trace:

$$\frac{1}{4} \sum_{\substack{r, r' \\ s, s'}} A_t^+ A_s = -\frac{e^4}{4St} \times$$

$$\text{Tr} \left[(\not{p}_1 + m) \gamma_\mu (\not{p}_2 - m) \gamma_\mu (\not{p}_4 - m) \gamma^\nu (\not{p}_3 + m) \gamma^\nu \right]$$

this gives trace of 8, 6 and 4 gamma matrices.

Simplification: go to the high-energy limit, where we can ignore the electron mass: $E_{cm} \gg m$.

then the most important term is the one which does not contain any m .

$$\frac{1}{4} \sum_t A_t^\dagger A_t \approx -\frac{e^4}{4st} \text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu \not{x}^\mu)$$

so let us compute this thing:

We want to bring two \not{x}^μ and \not{x}_μ close so we can contract them and use:

$$\begin{aligned} \not{x}^\mu \not{x}_\mu &= g^{\mu\nu} \not{x}_\mu \not{x}_\nu = \frac{1}{2} g^{\mu\nu} \{\not{x}_\mu, \not{x}_\nu\} \\ &= g^{\mu\nu} g_{\mu\nu} \mathbb{1} = 4\mathbb{1}. \end{aligned}$$

Also, notice that: $\{\not{x}_\mu, \not{x}^\nu\} = 2g_{\mu\nu}$

$$\begin{aligned} \text{and } \not{x}^\mu \not{x}_\mu &= -\underbrace{\not{x}^\mu \not{x}_\mu}_{4\mathbb{1}} \not{x}^\nu + \not{x}^\mu \underbrace{2g_{\mu\nu}}_{2\not{x}^\nu} \\ &= -2\not{x}^\nu \end{aligned}$$

So:

$$\begin{aligned} \text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu \not{x}^\mu) &= -\text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu \not{x}^\mu) \\ &\quad + 2g_{\mu\nu} \text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu) \\ &= +\text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu \not{x}^\mu) - 2g^{\mu\nu} \text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu) \\ &\quad + 2\text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_3 \not{x}_4 \not{x}^\nu) = \\ &= -2\text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_4 \not{x}_3^\nu \not{x}^\mu) - 2\text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_\mu \not{x}_4 \not{x}_3^\nu) \\ &\quad + 2\text{Tr}(\not{x}_1 \not{x}_v \not{x}_2 \not{x}_3 \not{x}_4 \not{x}^\nu) = \end{aligned}$$

$$\begin{aligned}
&= -2 \text{Tr}(\not{p}_1 \gamma_\nu \not{p}_2 \{ \not{p}_4, \gamma^\nu \} \not{p}_3) \\
&\quad - 4 \text{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) \qquad \{ \not{p}_4, \gamma^\nu \} = 2 p_4^\nu \\
&= -4 \text{Tr}(\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_3) - 4 \text{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) \\
&= -4 \left[\text{Tr}(\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_3 + \not{p}_4 \not{p}_1 \not{p}_2 \not{p}_3) \right] \\
&= -4 \text{Tr}(\{ \not{p}_1, \not{p}_4 \} \not{p}_2 \not{p}_3) \qquad \{ \not{p}_1, \not{p}_4 \} = 2 p_1 \cdot p_4 \\
&= -8 p_{14} \text{Tr}(\not{p}_2 \not{p}_3) = -32 p_{14} p_{23}
\end{aligned}$$

so, in the massless limit:

$$\frac{1}{4} \sum_{\text{spins}}^+ \mathcal{A}_t \mathcal{A}_s = \frac{8e^4}{st} p_{14} p_{23}$$

Similarly:

$$\frac{1}{4} \sum_{\text{spins}}^+ \mathcal{A}_s \mathcal{A}_t = \frac{8e^4}{st} p_{14} p_{23}$$

Overall, in the massless limit:

$$\frac{1}{4} \sum_{\text{spins}} |A|^2 = \frac{1}{4} \left[\sum |A_s|^2 + \sum |A_t|^2 + \sum A_s^\dagger A_t + \sum A_t^\dagger A_s \right]$$
$$= 8e^4 \left[\frac{P_{14} P_{23} + P_{13} P_{24}}{S^2} + \frac{P_{14} P_{23} + P_{12} P_{34}}{t^2} + 2 \frac{P_{14} P_{23}}{tS} \right]$$

~~$\frac{P_{14} P_{23}}{tS}$~~

Now recall that:

$$S = (P_1 + P_2)^2 - (P_3 + P_4)^2 = 2m^2 + 2P_{12} = 2m^2 + 2P_{34}$$
$$\approx 2P_{12} \approx 2P_{34} \quad (\text{in the } m^2 \rightarrow 0 \text{ limit})$$

Similarly:

$$t = (P_1 - P_3)^2 = (P_1 - P_4)^2 \approx -2P_{13} = -2P_{14}$$
$$u = (P_1 - P_4)^2 = (P_2 - P_3)^2 \approx -2P_{14} \approx -2P_{23}$$

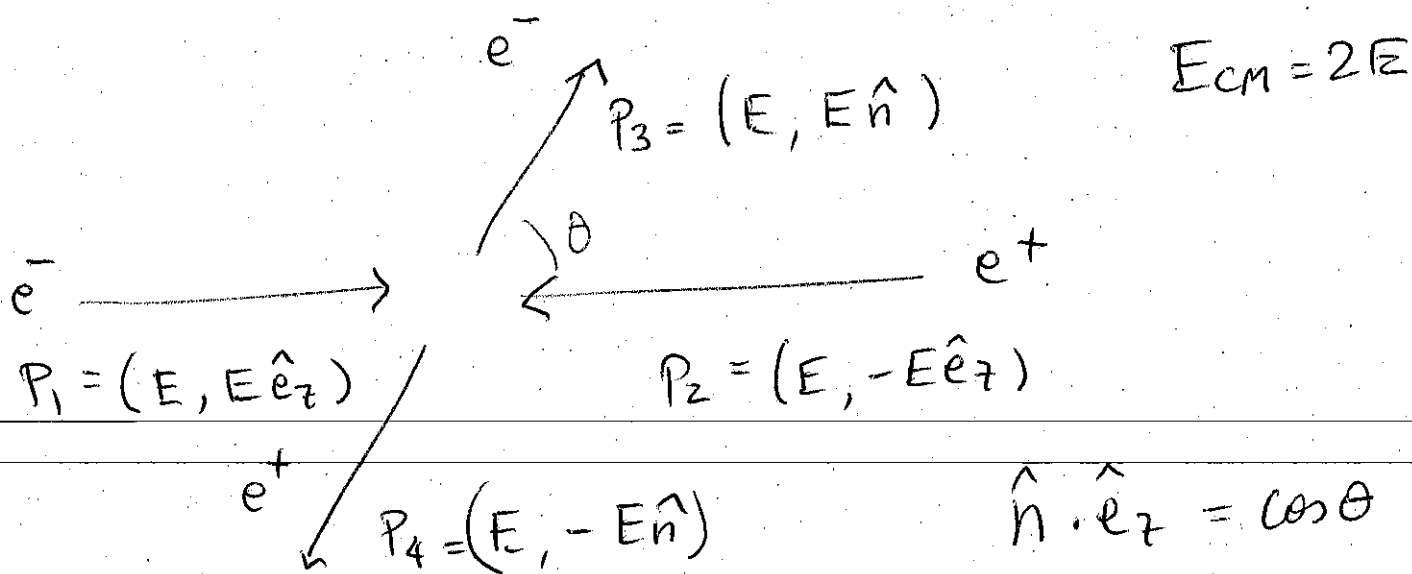
So the parenthesis is:

$$\left[\frac{u^2 + t^2}{S^2} + \frac{u^2 + S^2}{t^2} + \frac{2u^2}{tS} \right]$$

and finally:

$$\frac{1}{4} \sum_{\text{spins}} |A|^2 = 8e^4 \left[u^2 \left(\frac{1}{s} + \frac{1}{t} \right)^2 + \left(\frac{t}{s} \right)^2 + \left(\frac{s}{t} \right)^2 \right]$$

Now let us express this in terms of E and $\cos\theta$ in the CM frame:



$$s = (P_1 + P_2)^2 = 2P_1 P_2 = 4E^2$$

$$t = (P_1 - P_3)^2 = -2P_1 P_3 = -2E^2(1 - \cos\theta)$$

$$u = (P_1 - P_4)^2 = -2P_1 P_4 = -2E^2(1 + \cos\theta)$$

so:

$$\frac{1}{4} \sum_{\text{spins}} |A|^2 = 8e^4 \left[(1 + \cos\theta)^2 \left(\frac{1}{2} - \frac{1}{(1 - \cos\theta)} \right)^2 + \frac{(1 - \cos\theta)^2}{4} + \frac{4}{(1 - \cos\theta)^2} \right]$$

Finally the cross section is:

$$\frac{d\sigma}{d\Omega} \Big|_{cm} = \frac{e^4}{8\pi^2} \frac{1}{E_{cm}} \left[(1+\cos\theta)^2 \left(\frac{1}{2} - \frac{1}{1-\cos\theta} \right)^2 + \frac{(1-\cos\theta)^2}{4} + \frac{4}{(1-\cos\theta)^2} \right]$$