QFT - Final Exam

1 Chiral symmetries and their breaking

Consider a theory with the following field contents: 1 complex scalar ϕ; 1 left-handed spinor ψ_L; 1 right-handed spinor χ_R; with the Lagrangian given by:

$$\begin{aligned}\mathcal{L} \quad &= (\partial_\mu \phi)^*(\partial^\mu \phi) \;+\; \epsilon\, \mu^2 |\phi|^2 \;-\; \frac{\lambda}{2}|\phi|^4 \\ &+ i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L \;+\; i\bar{\chi}_R \gamma^\mu \partial_\mu \chi_R \;+\; g\left(\phi\, \bar{\psi}_L \chi_R \;+\; \phi^*\, \bar{\chi}_R \psi_L\right),\end{aligned} \tag{1}$$

where $\epsilon = \pm 1$, μ and g are real constants and $\lambda > 0$. We define two independent $U(1)$ transformations, (which we will call $U(1)_L$ and $U(1)_R$):

$$U(1)_L : \begin{cases} \psi_L \to e^{i\alpha}\psi_L \\ \chi_R \to \chi_R \\ \phi \to e^{i\alpha}\phi \end{cases} , \quad U(1)_R : \begin{cases} \psi_L \to \psi_L \\ \chi_R \to e^{i\beta}\chi_R \\ \phi \to e^{-i\beta}\phi \end{cases} \tag{2}$$

where α and β are real parameters.

1.1 Global case

1. Show that the $U(1)_L \times U(1)_R$ transformations (2) are a symmetry of the Lagrangian (1), and find the corresponding Noether currents.
2. Rewrite the Lagrangian, the action of the $U(1)_L \times U(1)_R$ symmetry, and the Noether currents in terms of a suitable Dirac spinor Ψ constructed out of ψ_L and χ_R.
3. Find the classical vacua of the theory and discuss their stability, in the two cases $\epsilon = 1$ and $\epsilon = -1$. *(Remark: in general, a spinor has to be zero in the classical vacuum if we want the latter to be Poincaré-invariant).*
4. In each of the above cases, discuss whether the $U(1)_L \times U(1)_R$ symmetry is spontaneously broken, and to which residual subgroup, in the stable vacuum.
5. For both values of ϵ, find the spectrum of excitations (by giving their mass and spin) around the stable vacuum.

1.2 Local case

Suppose now we want to make the $U(1)_L \times U(1)_R$ symmetry local. We will assume that the two corresponding gauge couplings are equal.

6. Introduce the appropriate gauge fields, specify their gauge transformations, and write the gauge-invariant generalisation of the Lagrangian (1)
7. For $\epsilon = \pm 1$, discuss the breaking of the gauge symmetry and give the new excitation spectrum (mass, spin and charges) around the stable vacuum.

2 Massive vector field

The dynamics of a massive spin-one field is described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

2.1 Field equations

1. Show that the action (3) is not invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

2. Write down the Euler-Lagrange equations

3. Show that, for $m \neq 0$, the Euler-Lagrange equations imply that A_μ must satisfy

$$\partial^\mu A_\mu = 0. \tag{4}$$

   Why is this *not* the same as a gauge-condition?

4. Using the EL equations, show that A_0 is not a dynamical degree of freedom but it can be eliminated at each instant of time (i.e. by solving an equation with no time-derivatives on A_0) in terms of the spatial components A_i.

5. Show that the spatial components A_i each satisfy Klein-Gordon's equation with mass m.

2.2 Propagator

We now want to write down the propagator for the massive spin-1 theory.

6. Show that, up to total derivatives, the Lagrangian (3) can be rewritten as

$$\mathcal{L} = \frac{1}{2}A^\mu \left(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu\right) A^\nu + \frac{1}{2}m^2 A_\mu A^\mu \tag{5}$$

7. We now add a source term to the Lagrangian, of the from

$$L_{source} = -J_\mu A^\mu.$$

   Write the field equation in the presence of the source, and the corresponding equation for the associated Green's function $G_{\mu\nu}(x, y)$.

8. By going to momentum-space, find the propagator $G_{\mu\nu}(p)$ *(Hint: by Lorentz-invariance, $G_{\mu\nu}(p)$ can only be the sum of two types of terms: $G_{\mu\nu}(p) = A(p^2, m)g_{\mu\nu} + B(p^2, m)p_\mu p_\nu$ where A and B are functions of p^2 and m to be determined.)*

9. Show that, on-shell (i.e. when $p^2 = m^2$), the propagator satisfies $p^\mu G_{\mu\nu}(p) = 0$, in accordance with the constraint equation (4).

2.3 Recovering Gauge invariance

We can make the theory of a massive spin-1 gauge-invariant by introducing an appropriate auxiliary scalar field $\pi(x)$, such that under a gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \qquad \pi(x) \to \pi(x) + \alpha(x).$$

10. Find a gauge-invariant Lagrangian for the fields $A_\mu(x)$ and $\pi(x)$ which is equivalent to the original Lagrangian (3).

11. Discuss the differences and similarities between this procedure and the Higgs mechanism for giving mass to A_μ.

3 Tree-level amplitudes in Yukawa theory

We consider a Yukawa-type theory for charged spin-1/2 particles (nucleons and anti-nucleons), described by a massive Dirac spinor ψ ,and neutral spin-0 particles (pions) described by a massive real scalar field φ. We take the Lagrangian to be:

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}\mu^2 \varphi^2 - \lambda \varphi \bar{\psi}\psi \tag{6}$$

where m is the nucleon mass, μ the pion mass, and λ a real Yukawa coupling constant.

3.1 Feynman rules

1. Write down the propagators for the nucleon and pion fields.

2. Draw the interaction vertex and give the associated value.

3. What is the dimension of the coupling constant λ? What is the type of the interaction (relevant, irrelevant, marginal)?

3.2 Feynman diagrams

Draw the tree-level Feynman diagrams which contribute to the following processes:

3. Nucleon-nucleon scattering, $\psi\psi \to \psi\psi$

4. Nucleon-anti-nucleon scattering, $\psi\bar{\psi} \to \psi\bar{\psi}$

5. Nucleon-pion scattering, $\psi\phi \to \psi\phi$.

3.3 Nucleon-nucleon scattering cross section

We will consider in detail the process of nucleon-nucleon scattering.

6. After specifying each of the external particles momentum and polarization, from the Feynman diagrams found above (there are two contributing to this process at tree level), write the scattering amplitude for nucleon-nucleon scattering. How is the relative sign between the two contributions fixed ?

7. For the unpolarized cross-section, compute the square of the amplitude, summed over final spins and averaged over initial spins. Show that the result is:

i. in the low energy limit $|\vec{p}_i| \ll m, \mu$:

$$\frac{1}{4}\sum_{spins}|\mathcal{A}|^2 \simeq 6\frac{\lambda^4 m^4}{\mu^4}; \tag{7}$$

ii. in the high energy limit $|\vec{p}_i| \gg m, \mu$:

$$\frac{1}{4}\sum_{spins}|\mathcal{A}|^2 \simeq \lambda^4\left\{(t^2+u^2)\left(\frac{1}{t^2}+\frac{1}{u^2}-\frac{1}{2tu}\right)+\frac{s^2}{2tu}\right\}; \tag{8}$$

where

$$s = (p_1+p_2)^2, \quad t = (p_1-p_3)^2, \quad u = (p_1-p_4)^2$$

(where p_1, p_2 are incoming momenta and p_3, p_4 are outgoing momenta). You may use the sum rules:

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

and the trace identities:

$$Tr(\mathbf{1}) = 4, \quad Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \quad Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).$$

8. Using the result (7), compute the scattering cross section in the center-of-mass frame and in the low-energy limit ($|\vec{p}_i| \ll m, \mu$).

3.4 Yukawa theory vs. four-fermion interaction

We want to compare the result of the Yukawa theory with those obtained in a theory where the interaction between the nucleons ψ is pointlike (i.e. not mediated by a meson φ). For this, we take the Lagrangian to be:

$$L_{4F} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{G_F}{2}(\bar{\psi}\psi)(\bar{\psi}\psi). \tag{9}$$

We will be interested in orders of magnitude estimates (no actual calculations of amplitudes will be demanded).

9. What is the dimension of the four-fermion coupling constant G_F? What is the type of the interaction (relevant, irrelevant, marginal)?

10. Draw the Feynman diagram(s) contributing to nucleon-nucleon scattering in the four-fermion theory described by the Lagrangian (9).

11. Using dimensional analysis, estimate the center-of-mass nucleon scattering cross section at fixed scattering angle and high-energy $E_{CM} \gg m$.

12. Use dimensional analysis and/or the result (8) to estimate the high-energy ($E_{CM} \gg m, \mu$) limit of the center-of-mass cross section (at fixed angle) in the Yukawa theory (6).

13. Compare the estimates in the two models, commenting in particular on the following points: which of the two theories is better behaved at high energy? What can you say about the regime of validity of the two models?

14. Now estimate the cross sections in the two models at low energy ($|\vec{p}_i| \ll \mu, m$) and show that, with a suitable identification of the coupling G_F in terms of λ and μ, the four-fermion theory gives (approximately) the same result as the Yukawa theory.