QFT - Td 5 - Symmetry breaking + Higgs

(sum over repeated indices is understood throughout)

1] $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ with transformation $\Phi' = U\Phi$

1.1) i. $\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{\lambda}{2}(\Phi^\dagger \Phi)^2$

~~Place~~ $\Phi^\dagger \Phi$ is invariant under $SU(2)$ since

$$\Phi^\dagger \Phi \longrightarrow \Phi^\dagger \underbrace{U^\dagger U}_{\mathbb{1}} \Phi = \Phi^\dagger \Phi$$

since U is constant (global symmetry) the same goes through for the kinetic term

ii. Infinitesimal $SU(2)$ transformation:

$$U = e^{i\epsilon^a \tau^a} \simeq \mathbb{1} + i\epsilon^a \tau^a \qquad \epsilon^a \ll 1$$

where ϵ^a are real number and

$\{\tau^a\}_{a=1,2,3}$ are $\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Phi \to U\Phi \simeq \Phi + i\epsilon^a \tau^a \Phi$$

$$\boxed{\delta\Phi = i\epsilon^a \tau^a \Phi}$$

[in components $\delta\Phi^i = i\epsilon^a \tau^{a\,i}{}_J \Phi^J$ (sum over J understood)

3. In general, for ~~a~~ complex fields:

$$\epsilon J^\mu = \sum_i{}' \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^i} \delta \Phi^i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger_i} \delta \phi^\dagger_i \right)$$

Here : $\dfrac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} = (\partial_\mu \Phi)^\dagger$, $\dfrac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger} = \partial_\mu \Phi$

and we have three generators τ^a and three parameters ~~3~~ $\epsilon^a \implies$ 3 currents J^a_μ

$a = 1, 2, 3$.

Doing a transformation only in the direction a we find:

$$\epsilon^a J^a_\mu = (\partial_\mu \Phi^\dagger) \delta_a \Phi + \delta_a \Phi^\dagger \partial_\mu \Phi =$$

$$= \partial_\mu \phi^\dagger (i \epsilon^a \tau^a) \phi + \phi^\dagger (-i \epsilon^a \tau^a) \partial_\mu \Phi$$

since τ^a are hermitian

$$= i \epsilon^a \left(\partial_\mu \Phi^\dagger \tau^a \phi - \phi^\dagger \tau^a \partial_\mu \Phi \right)$$

$$\to \boxed{J^a_\mu = i \left(\partial_\mu \Phi^\dagger \tau^a \Phi - \phi^\dagger \tau^a \partial_\mu \Phi \right)}$$

1.2)

1. $m^2 > 0$

i. We have to solve $\frac{\partial V}{\partial \varphi_i} = 0$

$$\frac{\partial V}{\partial \varphi_i} = \frac{\partial V}{\partial(\phi^\dagger\phi)} \frac{\partial \phi^\dagger\phi}{\partial \varphi_i} = V'(\phi^\dagger\phi)\,\varphi_i^* \qquad i=1,2$$

so either $V' = 0$ or $\varphi_1 = 0 = \varphi_2$

$V' = m^2 + \lambda \Phi^\dagger\Phi > 0$ (strictly) for $m^2 > 0$

$\Rightarrow \Phi_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only classical vacuum. It is stable because it is a minimum $(V \geq 0)$

ii. No, since $U\Phi_0 = 0 = \Phi_0$ therefore any SU(2) transformation leaves the vacuum invariant (no symmetry breaking)

iii. It is what one reads from the Lagrangian: 4 real scalars all with the same mass m^2.

2. $m^2 < 0$

i. As before, they are the solutions of

$$V'(\phi^\dagger\phi)\,\varphi_i^* = 0$$

- there is still a symmetry-preserving vacuum at $\varphi_i = 0$
- Now $V' = 0$ has a solution

$$\Phi_0^\dagger\Phi_0 = -\frac{m^2}{\lambda} > 0$$ ~~so any~~ any vector of the form $\begin{pmatrix}\varphi_1^0\\ \varphi_2^0\end{pmatrix}$ such that

$$\varphi_1^0\varphi_1^{0*} + \varphi_2^0\varphi_2^{0*} = -\frac{m^2}{\lambda}$$

is a vacuum.

these are the <u>stable</u> ones since now the vacuum at the origin is a <u>maximum</u>; on the other hand:

$$\cancel{\frac{\partial^2 V}{\partial\varphi_i^2}} = \left.\frac{\partial^2 V}{\partial\varphi_i\,\partial\varphi_i^*}\right|_{\Phi_0} = V''(\phi_0^\dagger\phi_0) = \lambda > 0$$

for the vacuum at $\phi_0^\dagger\Phi_0 \neq 0$.

ii. taking a generic vacuum:

$$\begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix} \longrightarrow U\begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix} \neq \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix}$$

so the symmetry is broken.

$$(U\Phi_0 \neq \Phi_0)$$ (however $U\Phi_0$ is <u>also</u> another <u>vacuum</u>)

To see whether there is a residual symmetry we can look at <u>infinitesimal</u> transformation, and choose one specific vacuum (the results will not depend on this choice) ~~take~~:

take $\Phi_0 = \begin{pmatrix} 0 \\ V/\sqrt{2} \end{pmatrix}$ $\quad V \in \mathbb{R}$, $\quad V = \sqrt{\frac{-2m^2}{\lambda}}$

$$\delta\Phi_0 = i\sum_a{}' \epsilon^a \tau^a \Phi_0 =$$

$$i\left[\frac{\epsilon^1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\frac{\epsilon^2}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + i\frac{\epsilon^3}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} 0 \\ V/\sqrt{2} \end{pmatrix}$$

$$= \frac{i}{2}\frac{V}{\sqrt{2}}\begin{pmatrix} \epsilon^1 - i\epsilon^2 \\ -\epsilon^3 \end{pmatrix} \neq 0$$ unless $\epsilon^1 = \epsilon^2 = \epsilon^3 = 0$ (they are <u>real</u>)

$$\Rightarrow \delta\Phi_0 \neq 0 \text{ unless } U = \mathbb{1}.$$

$\Rightarrow$ $SU(2)$ symmetry is completely broken since no generator (or combination of generators) leaves the vacuum invariant.

iii Now one has to change variables so that the fields are deviations from the vacuum: for example

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1(x) + i\varphi_2(x) \\ (v + h(x))\, e^{i\varphi_3(x)} \end{pmatrix}$$

and write $\mathcal{L}$ as a function of φ_1, φ_2, φ_3 and h.

A better choice is the following:

$$\Phi(x) = e^{i\sigma^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h(x)) \end{pmatrix}$$

where $\sigma^a(x)$ are the 3 real fields (notice that $\sigma^a(x) =$ cost corresponds to an $SU(2)$ rotation of the ground state, so the $\sigma^a(x)$ parametrise the direction that connect the various vacua $-$)

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \phi) - V(\phi^\dagger \phi)$$

- $\phi^\dagger \phi = \frac{1}{2}\left(V + h(x)\right)^2$ independent of $\sigma^a(x)$

so V does not contain $\sigma^a(x)$.

- $$(\partial_\mu \Phi) = e^{i\sigma^a \tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial_\mu h(x) \end{pmatrix} + \left(\partial_\mu \left(e^{i\sigma^a \tau^a}\right)\right) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(V+h) \end{pmatrix}$$

- $$(\partial_\mu \phi)^\dagger = \left(0, \frac{1}{\sqrt{2}} \partial_\mu h\right) e^{-i\sigma^a \tau^a} + \left(0, \frac{1}{\sqrt{2}}(V+h)\right)\left(\partial_\mu e^{-i\sigma^a \tau^a}\right)$$

$$(\partial_\mu \phi)^\dagger (\partial^\mu \phi) = \left(0, \frac{1}{\sqrt{2}} \partial_\mu h\right) e^{-i\sigma^a \tau^a} e^{i\sigma^a \tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial_\mu h \end{pmatrix}$$
$$+ \left(0, \frac{1}{\sqrt{2}} \partial_\mu h\right) e^{-i\sigma^a \tau^a} \partial_\mu e^{i\sigma^a \tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(V+h) \end{pmatrix}$$
$$+ \left(0, \frac{1}{\sqrt{2}}(V+h)\right)\left(\partial_\mu e^{-i\sigma^a \tau^a}\right) e^{i\sigma^a \tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial^\mu h \end{pmatrix}$$
$$+ \left(0, \frac{1}{\sqrt{2}}(V+h)\right) \partial_\mu\left(e^{-i\sigma^a \tau^a}\right) \partial^\mu\left(e^{i\sigma^a \tau^a}\right) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(V+h) \end{pmatrix}$$

- the first term is $\frac{1}{2}\partial_\mu h \partial^\mu h$

- the second and third terms cancel because:

$$e^{-i\sigma^a \tau^a} \partial_\mu e^{i\sigma^a \tau^a} + \left(\partial_\mu e^{-i\sigma^a \tau^a}\right) e^{i\sigma^a \tau^a}$$

$$= \partial_\mu \left(e^{-i\sigma^a \tau^a} e^{i\sigma^a \tau^a} \right)$$

$$= \partial_\mu \left(\mathbb{1} \right) = 0$$

- the ~~third~~ fourth term is:

$$\left(0, \frac{1}{\sqrt{2}}(v+h)\right) \partial_\mu \mathcal{U}^\dagger(x) \partial^\mu \mathcal{U}(x) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+h) \end{pmatrix}$$

where $\mathcal{U}(x) = e^{i\sigma^a(x)\tau^a}$

to analyze the spectrum let us expand everything to quadratic order: it is enough to keep linear terms in $\mathcal{U}(x)$ since

$$\partial_\mu \mathcal{U} \simeq \partial_\mu \left(1 + i\sigma^a(x)\tau^a\right) = (\partial_\mu \sigma^a)\tau^a$$

$$\partial_\mu \mathcal{U}^\dagger \partial^\mu \mathcal{U} \sim (\partial\sigma)^2 \text{ is quadratic}$$

then ; we can also drop h from the 4th term;

$$\left(0\,,\ \tfrac{1}{\sqrt{2}}v\right)(\partial_\mu\sigma^a)\tau^a(\partial^\mu\sigma^b)\tau^b\begin{pmatrix}0\\ \tfrac{1}{\sqrt{2}}v\end{pmatrix}$$

$$= \partial_\mu\sigma^a\partial^\mu\sigma^b\left(0\ \tfrac{1}{\sqrt{2}}v\right)\tau^a\tau^b\begin{pmatrix}0\\ \tfrac{1}{\sqrt{2}}v\end{pmatrix}$$

now use the fact that

$$\tau^a\tau^b = \frac{1}{4}\sigma^a\sigma^b \qquad \text{and} \quad \cancel{\sigma^a\sigma^b = \delta^{ab}\mathbb{1}}$$

$$\sigma^a\sigma^b = \delta^{ab}\mathbb{1}_2 + i\epsilon^{abc}\sigma^c$$

since $\partial_\mu\sigma^a\partial^\mu\sigma^b$ is symmetric in ab, the term with ϵ^{abc} does not contribute

$$= \partial_\mu\sigma^a\partial^\mu\sigma^b\left(0\ \tfrac{1}{\sqrt{2}}v\right)\frac{1}{4}\delta^{ab}\mathbb{1}_2\begin{pmatrix}0\\ \tfrac{1}{\sqrt{2}}v\end{pmatrix}$$

$$= \frac{1}{8}v^2\,\partial_\mu\sigma^a\partial^\mu\sigma^a$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{1}{8}v^2\,\partial_\mu\sigma^a\partial^\mu\sigma^a +$$

$$\underbrace{+\,V_0 - \frac{m^2}{2}h^2 - \frac{\lambda}{2}\left(\frac{v^2h^2}{4}\times 6\right)}$$

potential expanded to quadratic order in h, where $V_0 = m^2v^2/2 + \lambda v^4/8$

using $m^2 = -\frac{\lambda v^2}{2}$ the final result is:

$$\boxed{\mathcal{L}^{(2)}(h,\sigma^a) = \frac{1}{2}\partial_\mu h \partial^\mu h + \sum_{a=1}^{3} \frac{1}{8} v^2 \partial_\mu \sigma^a \partial^\mu \sigma^a + V_0 - \frac{1}{2}\lambda v^2 h^2}$$

- we see:

 1 <u>massive</u> real scalar h
 with mass $\mu^2 = \lambda v^2$

 3 <u>massless</u> real scalars $\sigma^1, \sigma^2, \sigma^3$
 (these are the goldstone bosons)

[total : 4 scalars
 as in the unbroken
 symmetry case]

1.3) - SU(2) gauge symmetry

1. $\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi^\dagger \phi) - \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$

where:

- there are 3 gauge ~~to~~ fields A_μ^a, which combine in a matrix:

$$A_\mu = A_\mu^a \tau^a = \frac{1}{2} \begin{pmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{pmatrix}$$

A_μ <u>real</u>

- the covariant derivative D_μ is:

$$D_\mu \Phi = \partial_\mu \Phi - ig A_\mu \Phi$$

(or in components:

$$(D_\mu \Phi)_i = \partial_\mu \Phi_i - ig A_\mu^a \tau_i^{a\, j} \Phi_j$$

- the field strenght is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]$$

2. For $m^2 < 0$ the stable vacuum is (see part 1.2): $\Phi_0^\dagger \Phi_0 = v^2/2$

Choose $\Phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$

to expand around the vacuum, choose unitary gauge (which can always be done by an $SU(2)$ local transformation on a generic vector $\Phi = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$):

$$\Phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h(x)) \end{pmatrix}$$

- the potential for $\Phi(x)$ has the same form as for the global case: to quadratic order:

$$V(\phi^\dagger \phi) = V_0 + \frac{\lambda v^2}{2} h^2 + O(h^3)$$

- the kinetic term is given as follows:

$$D_\mu \Phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial_\mu h \end{pmatrix} - i g A_\mu^a \tau^a \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+h) \end{pmatrix}$$

$$(D_\mu \Phi)^\dagger = \left(0, \frac{1}{\sqrt{2}} \partial_\mu h\right) + i g A_\mu^a \left(0 \;\; \frac{1}{\sqrt{2}}(v+h)\right) \tau^a$$

since the second term in each line is already linear in A_μ, and we are going to square it, if we are only interested in the Lagrangian up to quadratic order we can drop h in the terms containing A_μ.

$$\Rightarrow (D_\mu \phi)^\dagger (D^\mu \phi) = \frac{1}{2} \partial_\mu h \partial^\mu h +$$

$$\underline{- ig \left(0, \frac{1}{\sqrt{2}} \partial_\mu h \right) \tau^a \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} A^{\mu a}}$$

$$\underline{+ ig A_\mu^a \left(0, \frac{1}{\sqrt{2}} v \right) \tau^a \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial^\mu h \end{pmatrix}}$$

$$+ g^2 \underbrace{A_\mu^a A^{\mu b}}_{\text{symmetric in } ab} \left(0, \frac{1}{\sqrt{2}} v \right) \underbrace{\tau^a \tau^b}_{\frac{1}{4}\delta^{ab} \mathbb{1} + \text{antisymmetric in } ab} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix}$$

$$= \frac{1}{2} \partial_\mu h \partial^\mu h$$

$$+ \underbrace{\frac{v^2 g^2}{8}}_{\frac{1}{2} m_A^2} A_\mu^a A^{\mu a} \longrightarrow$$ All 3 gauge bosons get the same mass $= v^2 g^2 / 4$

- the quadratic part of the gauge field kinetic term is:

$$-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} = -\frac{1}{2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})$$

(same as 3 independent Maxwell fields.

Spectrum :

- 1 Real scalar h with mass $m_h^2 = \lambda v^2$
- 3 massive vectors A_μ^a with mass $m_A^2 = \frac{1}{4} v^2 g^2$

(- no massless particles)

2

2.1) A vector transformation acts as

$$\vec{\phi} \to O\vec{\phi} \quad \text{where } O \in SO(3)$$
$$({}^tO O = \mathbb{1}, \ \det O = 1)$$

For an infinitesimal transformation,

$$O \simeq \mathbb{1} + \epsilon^a T^a \qquad a = 1, 2, 3$$

where T^a generate real anti-symmetric matrices $({}^tT^a = -T^a)$

we can take : $(T^a)^{bc} = \epsilon^{abc}$, i.e.

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad T^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

notice that $[T^a, T^b] = \epsilon^{abc} T^c$

[so if we wanted <u>hermitian</u> generators we should multiply by i : $\tilde{T}^a = i T^a$ are hermitian and satisfy the $SU(2)$ algebra :
$$[\tilde{T}^a, \tilde{T}^b] = i \epsilon^{abc} \tilde{T}^c \quad]$$

The infinitesimal transformation is then:

$$\vec{\Phi}' = \vec{\Phi} + \epsilon^a T^a \Phi + O(\epsilon^2)$$

or $$\delta \Phi = \epsilon^a T^a \Phi$$ or, in components:

$$\boxed{\delta \Phi^a = -\epsilon^{abc} \epsilon^b \phi^c}$$

[Alternatively, we could have arranged $\vec{\phi}$ in a 2×2 hermitian traceless matrix:

$$\Phi = \frac{1}{2} \begin{pmatrix} \phi^3 & \phi^1 - i\phi^2 \\ \phi^1 + i\phi^2 & -\phi^3 \end{pmatrix} = \Phi^a \tau^a$$

then the transformation acts as the adjoint action of SU(2):

$$\Phi \to U \Phi U^{-1}$$

Infinitesimally, $U = e^{i\omega}$ $\quad \omega = \epsilon^a \tau^a$

$$\Phi \to \Phi + i\omega\Phi - i\Phi\omega + O(\omega^2)$$

$$\Rightarrow \delta\Phi = i[\omega, \Phi]$$

In components:

$$\delta\Phi = i\, \epsilon^b \Phi^c [\tau^b, \tau^c] = -\epsilon^{abc} \epsilon^b \Phi^c \tau^a$$

$$\Rightarrow \delta\Phi^a = -\epsilon^{abc}\epsilon^b \Phi^c$$ same as before!

]

2. As before: three currents $J^{\mu a}$

$$\epsilon^a J^{\mu a} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^b} \delta \Phi^b = \partial_\mu \phi^b \delta \phi^b$$

$$= -\partial_\mu \phi^b \epsilon^{bcd} \epsilon^c \phi^d$$

Replacing c with a and getting rid of ϵ

$$\Rightarrow \quad J^{\mu a} = + \epsilon^{abc} \partial_\mu \phi^b \phi^c$$

$$= + \frac{1}{2} \epsilon^{abc} (\partial_\mu \phi^b \phi^c - \partial_\mu \phi^b \phi^c)$$

$$= \frac{1}{2} {}^t(\partial_\mu \vec{\phi}) T^a \vec{\phi} - \frac{1}{2} {}^t\vec{\phi} T^a \partial_\mu \vec{\phi}$$

$$= \left(\text{or just} = - {}^t\vec{\phi} T^a \partial_\mu \vec{\phi} \right)$$

2,2) Same thing as for the doublet:

$$\frac{\partial V}{\partial \phi^a} = 0 \Rightarrow V'(|\phi|^2) \phi^a = 0 \quad \forall a$$

for $m^2 < 0$ $V' = 0$ at $\vec{\phi}_0 = 0$ (maximum)

or $|\vec{\phi}_0|^2 = -\frac{m^2}{\lambda}$ (minimum)

stable vacuum: $\vec{\Phi}_0$ such that $\boxed{|\vec{\Phi}_0| = \sqrt{-\frac{m^2}{\lambda}}}$

Pick a particula vacum, eg.

$$\vec{\Phi}_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \qquad v \equiv \sqrt{\frac{-m^2}{\lambda}}$$

this is invariat under rotations in the 3-direction: take:

$$O = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{R_2} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \qquad \text{(Rotation about the z-axis)}$$

$$O \vec{\Phi}_0 = \vec{\Phi}_0$$

So the subgroup of rotations in the 12 plane is a residual symmetry of the vacum: $SO(3)$ is broken to

$$SO(2)$$

or, in 2×2 complex notation: $\Phi_0 = \frac{1}{2}\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$

$$SU(2) \longrightarrow U(1)$$

the $U(1)$ is generated by $\tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Indeed: $\cancel{\delta_3^3 \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}} = \epsilon \tau^3,$

$$\delta \Phi_0 = \epsilon^3 [\tau^3, \Phi^0] = 0$$

[The same result is true whatever choice of the vacuum we had made: the general vacuum is

$$\Phi_{\vec{n}} = v\,\vec{n} \quad \text{with } |\vec{n}| = 1$$

then rotations around the $\vec{n}$ axis would leave $\Phi_{\vec{n}}$ invariant (still $SO(2)$ group)]

2. We want to study deviations from the vacuum and write $\mathcal{L}$ to quadratic order in these deviations.

To exploit invariance under rotations, write:

$$\Phi(x) = \mathcal{O}(x) \begin{pmatrix} 0 \\ 0 \\ v + h(x) \end{pmatrix}$$

where $\mathcal{O}(x)$ is any combination of rotation around the 1 and 2 axes but <u>not</u> around the 3-axis:

$$\mathcal{O}(x) = e^{\frac{i}{v}\left(\sigma^1(x) T^1 + \sigma^2(x) T^2\right)}$$

- the potential term does not contain $\sigma^1(x)$ and $\sigma^2(x)$: to quadratic order:

$$V = \frac{m^2}{2}(v+h)^2 + \frac{\lambda}{4}(v+h)^4$$

$$\simeq V_0 + \frac{m^2}{2}h^2 + \frac{\lambda}{4}6v^2h^2 + O(h^3)$$

$$\simeq V_0 + \frac{1}{2}(2v^2\lambda)h^2 + O(h^3)$$

- For the kinetic term, the calculation is similar as in the SU(2) case with $U(x)$ replaced by $O(x)$:

$$\frac{1}{2}{}^t(\partial_\mu\phi)(\partial^\mu\phi) = \frac{1}{2}\partial_\mu h\,\partial^\mu h \quad +$$

$$\frac{1}{2}(0\ ,\ 0\ ,\ v+h)\ \partial_\mu {}^t O\ \partial^\mu O \begin{pmatrix}0\\0\\v+h\end{pmatrix}$$

to quadratic order

$$= \frac{1}{2}\partial_\mu h\,\partial^\mu h \ +$$

$$\frac{1}{2}(0,0,v)\left[\partial_\mu\sigma^1\,{}^tT^1 + \partial_\mu\sigma^2\,{}^tT^2\right]\left[\partial^\mu\sigma^1 T^1 + \partial^\mu\sigma^2 T^2\right]\begin{pmatrix}0\\0\\v\end{pmatrix}$$

only the 33 component of the matrix product contribute, and:

$$({}^tT^1 \, T^1)^{33} = 1$$

$$({}^tT^2 \, T^2)^{33} = 1$$

$$({}^tT^1 \, T^2)^{33} = ({}^tT^2 \, T^1)^{33} = 0$$

$$= \frac{1}{2} \partial_\mu h \, \partial^\mu h + \frac{1}{2} V^2 \partial_\mu \sigma^1 \partial^\mu \sigma^1 + \frac{1}{2} V^2 \partial_\mu \sigma^2 \partial^\mu \sigma^2$$

In the end: the quadratic term are: $(m_h^2 = 2V^2\lambda)$

$$\boxed{\mathcal{L}^{(2)} = \frac{1}{2} \partial_\mu h \, \partial^\mu h - \frac{1}{2} m_h^2 h^2 + \frac{1}{2} V^2 \partial_\mu \sigma^1 \partial^\mu \sigma^1 + \frac{1}{2} V^2 \partial^\mu \sigma^2 \partial_\mu \sigma^2}$$

1 massive scalar with $(\text{mass})^2 = 2V^2\lambda$

2 massless scalars σ^1, σ^2 (Goldstone bosons)

$$\boxed{\begin{array}{l}\#\ \text{Goldston bosons} \\ = \#\ \text{Broken symmetry generators} \\ \quad (\text{in this case, } 2)\end{array}}$$

2.3) Again introduce 3 vector fieds
1. A_μ^a. Now the covariant derivative is:

$$D_\mu \vec{\Phi} = \partial_\mu \vec{\phi} + g A_\mu^a T^a \vec{\phi}$$

(everything is real, so no i, but T^a are not hermitian but antisymmetric

$$= \partial_\mu \vec{\phi} - i g A_\mu^a \tilde{T}^a \vec{\phi}$$

(now it is written in the usual way with $\tilde{T}^a = i T^a$ hermitian)

$$\mathcal{L} = \frac{1}{2} {}^t(D_\mu \vec{\phi})(D^\mu \vec{\phi}) - \frac{1}{2} m^2 |\vec{\phi}|^2 - \frac{\lambda}{4} |\vec{\phi}|^4$$
$$- \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu}$ is as before
(in particular we can still write

$F_{\mu\nu} = F_{\mu\nu}^a \tau^a$ with $\tau^a = \frac{1}{2} \sigma^a$ in the <u>fundamental</u> representation of $SU(2)$.)

2. the vacuum is still $\bar{\Phi}_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

Now the fluctuation in the $\sigma^1(x)$ and $\sigma^2(x)$ direction can be rotated away by a local $SO(3)$ transformation, so ~~the~~ we set

$$\Phi(x) = \begin{pmatrix} 0 \\ 0 \\ v+h(x) \end{pmatrix}$$

- the potential is as before,

$$V \simeq V_0 + \frac{1}{2} m_h^2 h^2 \qquad m_h^2 = 2v^2\lambda$$

- the kinetic term is; to quadratic order:

$$\frac{1}{2}(D_\mu \phi)^t (D^\mu \phi) = \frac{1}{2} \partial_\mu h \, \partial^\mu h \; +$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & v \end{pmatrix} g^2 A_\mu^a T^{t\,a} A^{\mu b} T^b \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$$= \frac{1}{2} g^2 A_\mu^a A^{\mu b} \begin{pmatrix} 0 & 0 & v \end{pmatrix} T^{t\,a} T^b \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

As before, only the 33 component contributes.

One can check easily that

$$({}^tT^1 T^1)^{33} = 1$$

$$({}^tT^2 T^2)^{33} = 1$$

$$({}^tT^3 T^3)^{33} = 0 \quad (!)$$

and the ones with $a \neq b$ also give zero

$\rightarrow$

$$\frac{1}{2}{}^t(D_\mu \phi)(D^\mu \phi) = \frac{1}{2}\partial_\mu h\, \partial^\mu h \; +$$

$$+ \; \frac{1}{2} g^2 v^2 A^1_\mu A^{\mu 1} + \frac{1}{2} g^2 v^2 A^2_\mu A^{\mu 2}$$

so there is <u>no</u> mass for A^3_μ : indeed

A^3_μ is associated with the <u>unbroken</u> symmetry generator T^3, so it stays massless the full quadratic Lagrangian is:

$$\mathcal{L}^{(2)} = \frac{1}{2}\partial_\mu h\, \partial^\mu h - \frac{1}{2} m_h^2 h^2 +$$
$$- \frac{1}{4}(\partial_\mu A_\nu^1 - \partial_\nu A_\mu^1)^2 + \frac{1}{2} g^2 v^2 A^{\mu 1} A_\mu^1$$
$$- \frac{1}{4}(\partial_\mu A_\nu^2 - \partial_\nu A_\mu^2)^2 + \frac{1}{2} g^2 v^2 A^{\mu 2} A_\mu^2$$
$$- \frac{1}{4}(\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3)$$

- 1 massive scalar, $m_h^2 = 2v^2\lambda$
- 2 massive vectors $A_\mu^{1,2}$ with mass $m_A^2 = g^2 v^2$
- 1 massless vector A_μ^3, which generate the residual $U(1)$ gauge symmetry.

Under this $U(1) \simeq SO(2)$:

- h is unchanged (neutral) (it is invariant by construction)
- A_μ^1 and A_μ^2 transform: indeed

$$\delta A_\mu^a = \partial_\mu \omega^a + g\epsilon^{abc} \omega^b A_\mu^c$$

If we take $\omega^3 = \alpha$, $\omega^1 = \omega^2 = 0$

(the unbroken gauge symmetry :

$$\begin{cases} \delta A_\mu^3 = \partial_\mu \alpha \\ \delta A_\mu^1 = -g\alpha A_\mu^2 \\ \delta A_\mu^2 = g\alpha A_\mu^1 \end{cases}$$

this is the same as a 2-component vector $\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$ transforms under rotations around X^3. We can make A_μ^1, A_μ^2 into a charged field :

$$A_\mu^+ = -A_\mu^1 - i A_\mu^2$$

$$A_\mu^- = -A_\mu^1 + i A_\mu^2$$

Now : $$\delta A_\mu^+ = i\alpha(x) g A_\mu^+$$
$$\delta A_\mu^- = -i\alpha(x) g A_\mu^-$$

which is the infinitesimal version of

$$A_\mu^+ \to e^{i\alpha(x)g} A_\mu^+ , \quad A_\mu^- \to e^{-i\alpha(x)g} A_\mu^-$$

so A^+_μ and A^-_μ transform under $U(1)$ as ~~two~~ a charge $= 1$ and a charge $= -1$ field. (in units of g)

$\Rightarrow$ they are charged vectors, with unit ~~ch~~ elementary charges $\pm g$.