M2 NPAC 2019/2020

QFT - part 2 **TD n°2**

Interacting field theories and the S-matrix

1. Dimensional Analysis

1. Consider a theory with one scalar field ϕ and one Dirac spinor ψ.
   a) Find the dimension of the coupling constants of the following interactions :
   $$g\,\phi\,\overline{\psi}\psi\;, \qquad G_F\,\overline{\psi}\psi\,\overline{\psi}\psi$$
   b) Is each of the interactions above relevant, marginal, or irrelevant ? For the irrelevant ones, what is the energy scale at which perturbation theory breaks down ?
   c) Write all possible interactions up to dimension six compatible with Lorentz invariance and parity.
2. In Einstein's theory of gravitation, the coupling constant is the same as Newton's gravitational constant G_N (in S.I. units : $G_N = 6.7\,10^{-11}\,m^3Kg^{-1}s^{-2}$).
   a) Find the dimension of G_N in natural units. Is this coupling relevant or irrelevant ?
   b) If the latter is true, find the value (in GeV) of corresponding strong coupling energy scale.

2. Yukawa cross-section in non-relativistic QM

In the Born approximation, calculate the differential cross-section in non-relativistic quantum mechanics for the scattering of a particle of mass m by a central potential of the Yukawa type :

$$V(\vec{r}) = \frac{\lambda}{4\pi}\frac{e^{-\mu r}}{r}, \qquad r \equiv |\vec{r}|. \tag{1}$$

where λ and μ are positive constants. What is the physical meaning of μ ?

3. Two-particle scattering in ϕ^4 theory

Recall that for relativistic scattering, the differential cross section can be written as

$$d\sigma = \frac{1}{4E_1E_2|}\frac{1}{|\vec{v}_1 - \vec{v}_2|}\int_{\vec{p}_3\to d\Omega} |\mathcal{A}_{i\to f}|^2 d\Pi_{LIPS}, \tag{2}$$

where the amplitude $\mathcal{A}_{i\to f}$ is defined by extracting a momentum-conservation *delta*-function from the S- matrix element,

$$\langle f|S-1|i\rangle = \mathcal{A}_{i\to f}\,\delta^{(4)}\left(\sum p_f - \sum p_i\right), \tag{3}$$

and the Lorentz Invariant Phase Space element is given by :

$$\Pi_{LIPS} = \delta^{(4)}(p_3+p_4-p_1-p_2)\frac{d^3p_3}{(2\pi)^3 2\omega_{p_3}}\frac{d^3p_4}{(2\pi)^3 2\omega_{p_4}} \tag{4}$$

Finally, recall that for a collision in the center of mass frame, and when all particles have the same mass, one obtains :

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|\mathcal{A}_{i\to f}|^2}{E_{CM}^2} \tag{5}$$

where E_{CM} is the total center-of-mass energy of the collision.

a. Warm up : one-dimensional harmonic oscillator

Consider a non-relativistic quantum harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2m}p^2 = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

with

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = \sqrt{\frac{m\omega}{2}}i(a - a^\dagger), \qquad [a, a^\dagger] = 1$$

and eigenstates $|n\rangle$ satisfying

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle \qquad (a|0\rangle = 0).$$

1. Compute the expectation values

$$\langle 0|x^2|0\rangle, \qquad \langle 0|x^4|0\rangle$$

2. Compute the matrix element

$$\langle 0|x^4|2\rangle$$

b. Cross section in ϕ^4 theory

We now consider ϕ^4 theory, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

1. Compute, to the lowest-order in λ, the S-matrix element for $2 \to 2$ scattering,

$$\langle p_1', p_2'|S - 1|p_1, p_2\rangle$$

For this purpose, use Dyson's formula for the S-matrix,

$$S = T\exp\left(-i\int dt H_I(t)\right)$$

as well as the free field mode expansion in terms of creation and annihilation operators,

$$\phi(x) = \phi^+(x) + \phi^-(x), \quad \phi^+(x) \equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} a_{\vec{p}}\, e^{-ipx}, \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} a_{\vec{p}}^\dagger e^{ipx},$$

and the commutation relation

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q}).$$

2. Recalling the definition (3), what is the scattering amplitude $\mathcal{A}_{i\to f}$?
3. From the result found above and from (5), obtain the differential cross-section for $2 \to 2$ scattering in ϕ^4 theory in the center-of-mass frame for the incoming particles.

Congratulations ! You have just computed your first QFT observable.