13.1 QED Feynman rules

The Feynman rules for QED can be read directly from the Lagrangian just as in scalar QED. The only subtlety is possible extra minus signs coming from anticommuting spinors within the time ordering. First, we write down the Feynman rules, then derive the supplementary minus sign rules.

A photon propagator is represented with a squiggly line:

 $$= \frac{-i}{p^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \qquad (13.10)$$

Unless we are explicitly checking gauge invariance, we will usually work in Feynman gauge, $\xi = 1$, where the propagator is

 $$= \frac{-ig_{\mu\nu}}{p^2 + i\varepsilon} \quad \text{(Feynman gauge)}. \qquad (13.11)$$

A spinor propagator is a solid line with an arrow:

 $$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}. \qquad (13.12)$$

The arrow points to the right for particles and to the left for antiparticles. For internal lines, the arrow points with momentum flow.

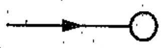
External photon lines get polarization vectors:

 $= \epsilon_\mu(p)$ (incoming), (13.13)

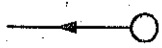
 $= \epsilon^\star_\mu(p)$ (outgoing). (13.14)

Here the blob means the rest of the diagram.

External fermion lines get spinors, with u spinors for electrons and v spinors for positrons.

 $= u^s(p)$, (13.15)

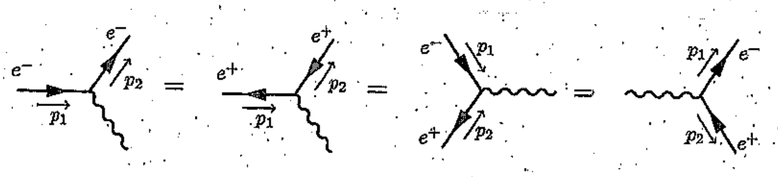
 $= \bar{u}^s(p)$, (13.16)

 $= \bar{v}^s(p)$, (13.17)

 $= v^s(p)$. (13.18)

External spinors are on-shell (they are forced to be on-shell by LSZ). So, for external spinors we can use the equations of motion:

$$(\not{p} - m)u^s(p) = \bar{u}^s(p)(\not{p} - m) = 0, \qquad (13.19)$$
$$(\not{p} + m)v^s(p) = \bar{v}^s(p)(\not{p} + m) = 0, \qquad (13.20)$$



$= -ie\gamma^\mu$.

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M2 NPAC 2019/2020

QFT **TD n°4**

QED

1 Spinor algebra

The Dirac matrices are defined in terms of the basic property :

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4 \tag{1}$$

where $g_{\mu\nu}$ is the Minkowski metric $diag(1, -1-1-1)$ and $\mathbf{1}_4$ is the identity matrix.

A basis for positive and negative frequency solutions of the Dirac equation is given by :

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\xi^s \\ \sqrt{p_\mu\bar{\sigma}^\mu}\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\xi^s \\ -\sqrt{p_\mu\bar{\sigma}^\mu}\xi^s \end{pmatrix}, \qquad s = 1, 2 \tag{2}$$

where ξ^s are the two-component spinors

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the expressions (2)

$$\sigma^\mu = (\mathbf{1}_2, \sigma^i), \qquad \bar{\sigma}^m u = (\mathbf{1}_2, -\sigma^i)$$

where σ^i are the Pauli matrices.

1.1 Traces of γ matrices

1. Without using the explicit representation of the γ-matrices, but only equation (1), show that

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad ,$$
$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \quad .$$

   where the trace is over the spinor indices.
2. Deduce $\mathrm{Tr}\,(\not{a}\,\not{b})$ et $\mathrm{Tr}\,(\not{a}\,\not{b}\,\not{c}\,\not{d})$.
3. Show that $\mathrm{Tr}\,(\gamma^\mu) = 0$ and that the same holds of the product of an odd number of γ-matrices.

1.2 Spin sums

In what follows, α, β... are spinor indices, and run from 1 to 4, s, s'... run over the polarisation (1,2).

1. Show that

$$\bar{u}^s(p)u^{s'}(p) = 2m\delta^{ss'}, \quad \bar{v}^s(p)v^{s'}(p) = -2m\delta^{ss'}, \quad \bar{v}^s(p)u^{s'}(p) = \bar{u}^s(p)v^{s'}(p) = 0$$

2. Show that

$$\bar{u}^s(p)\gamma^\mu u^{s'}(p) = 2\delta^{ss'}p^\mu$$

3. Show that

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

2 QED cross sections

The goal of this exercise is to calculate the unpolarized differential cross section for two simple QED processes, at tree level, in the center of mass frame. The result will be expressed as a function of the center of mass energy E_{CM} and the scattering angle θ (i.e. the angle between the outgoing particles and the incoming direction. The latter may be taken to be the z direction).

Recall that, for $2 \to 2$ scattering, the differential cross section in the center of mass is related to the amplitude by (cfr. TD3)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{CM}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{A}|^2 \tag{3}$$

where $\mathcal{A}$ is the scattering amplitude and $\vec{p}_{i,f}$ are the initial and final momenta of one of the particles.

2.1 e^+e^- (Bhabha) Scattering

Consider the process

$$e^+e^- \longrightarrow e^+e^-$$

1. Draw the tree-level Feynman diagrams which contribute to this process *(Hint : there are two of them : one in the s-channel, one in the t-channel).*
2. Find scattering amplitude associated to each of diagram. What is their relative sign ?
3. Compute the square of the amplitude using the spin sum rules, and the corresponding differential cross section using equation (3). Show that, in the high-energy limit $E_{cm} \gg m_e$, one finds :

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left[u^2 \left(\frac{1}{s} + \frac{1}{t} \right)^2 + \left(\frac{t}{s} \right)^2 + \left(\frac{s}{t} \right)^2 \right]. \tag{4}$$

   where s, t, u are the Mandelstam variables *(Notice that, if we ignore the electron mass, then $s + t + u = 0$).*
4. Rewrite equation (4) in terms of $\cos\theta$ and the center-of-mass energy.

2.2 Pair annihilation into photons

Consider the process

$$e^+e^- \longrightarrow \gamma\,\gamma$$

in the center-of-mass frame.

1. Draw the tree-level Feynman diagrams which contribute to this process *(Hint : there are two of them : one in the s-channel, one in the t-channel).*
2. Find scattering amplitude associated to each of diagram. What is their relative sign ?
3. Prove the *photon polarisation sum* rules :

$$\sum_{i=1}^{2} (\epsilon_\mu^i)^* \epsilon_\nu^i = -g_{\mu\nu} + \frac{1}{2E^2} (p_\mu \bar{p}_\nu + \bar{p}_\mu p_\nu) \tag{5}$$

   where $p_\mu = (E, \vec{p})$, $\bar{p}_\mu = (E, -\vec{p})$, ϵ^i with $i = 1, 2$ are two transverse polarizations (i.e. orthogonal to both p_μ and $\bar{p}_\mu$ (Notice that if p_μ is a null vector, so is $\bar{p}_\mu$. For example, $p_\mu = (E, 0, 0, E)$, $\bar{p}_\mu = (E, 0, 0, -E)$, $\epsilon_\mu^1 = (0, 1, 0, 0)$, $\epsilon_\mu^2 = (0, 0, 1, 0)$.)

4. Show that the amplitude vanishes whenever the polarisation is along p_μ or $\bar{p}_\mu$. Deduce that we can substitute

$$\sum_{i=1}^{2}(\epsilon^i_\mu)^*\epsilon^i_\nu \to -g_{\mu\nu}$$

when squaring the amplitude.

5. Compute the square of the amplitude using the result above *(make sure you first add the contribution from the two diagrams, then square).* and show that the corresponding differential cross section is :

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\frac{E}{p}\left[\frac{E^2+m^2+p^2\cos^2\theta}{m^2+p^2\sin^2\theta} - \frac{2m^4}{(m^2+p^2\sin^2\theta)^2}\right] \tag{6}$$

6. Show that, in the high-energy limit $E \gg m$, and for finite θ (i.e. for $\theta \gtrsim m/p$) equation (6) becomes

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)^2 \tag{7}$$