M2 NPAC 2018/2019

QFT - part 2 TD n°1

Green's functions and propagators

Useful relations :

$$\int_{-\infty}^{+\infty} dy\, e^{iyz} = 2\pi\delta(z); \qquad \frac{d}{dz}\theta(z) = \delta(z)$$

Cauchy's theorem :

$$\oint dz \frac{f(z)}{z-w} = 2n\pi i f(w)$$

where n is the number of times the integration contour goes around w anti-clockwise.

1. KG equation coupled to an external source

Consider the equation for a real scalar field with an external source :

$$(\Box + m^2)\phi(\vec{x},t) = J(\vec{x},t). \tag{1}$$

1. Assuming there is an instant t_0 such that $J(\vec{x}, t < t_0) = 0$, write, in terms of an appropriate Green's function, the solution $\phi(\vec{x},t)$ which satisfies $\phi(\vec{x}, t < t_0) = 0$. Which is the correctn Green's function in this case ?
2. Find the explicit solution $\phi(\vec{x},t)$ in the particular case :

$$J(\vec{x},t) = j_0\, \theta(t)\, e^{-\mu t}, \qquad j_0, \mu > 0, \qquad \theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$

3. Recall the solution for the *free* Klein-Gordon (1) :

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \Big(a(\vec{k})\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + a^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t} \Big) \tag{3}$$

where $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$. Show that, for $t \to \infty$, the solution found in the previous point reduces to a solution of the free equation with coefficients :

$$a(\vec{k}) = -\frac{(2\pi)^{3/2} j_0}{(m + i\mu)\sqrt{2m}}\, \delta^3(\vec{k}). \tag{4}$$

2. The propagator as a Green function

By explicit calculation applying the differential KG operator, show that the Feynman Green's function

$$G_F(x - x') = i\Big(\theta(t - t')D_+(x - x') + \theta(t' - t)D_-(x - x')\Big)$$

is a Green's function for the Klein-Gordon equation, where D_+ and D_- are the positive and negative freequency solutions,

$$D_\pm(x) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{\mp i(\omega_p t - \vec{p}\cdot\vec{x})}}{2\omega_p}$$

and $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$.

3. Wick's theorem

Prove, by induction, Wick's theorem,

$$T\phi(x_1)\ldots\phi(x_n) =: \phi(x_1)\ldots\phi(x_n) : + : \text{all possible contractions} :$$

where : : denotes normal ordering and $T\ldots$ denotes time-ordering and a contraction means replacing a pair of fields $\ldots\phi(x_a)\ldots\phi(x_b)\ldots$ the Feynman propagator,

$$\Delta_F(x_a - x_b) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x_a - x_b)} \frac{i}{p^2 - m^2 + i\epsilon}$$