Spherically symm metrics

- ds^2 can only depend on rotational invariants
$\left(t, r, d\vec{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \;\&\; \vec{x}\cdot d\vec{x} = r\,dr \right)$

$\Rightarrow$ ds^2 takes form in ④:

$$ds^2 = -C(t,r)\,dt^2 + D(t,r)\,dr^2 + 2E(t,r)\,dr\,dt + F(t,r)\,r^2\,d\Omega^2$$

Aim: show that can set $E=0$ & $F=1$ by changes of coord.

1) let $r' = r\,F(t,r)$ which can invert $\to r(t,r')$
so that $C(t,r) \to C'(t,r')$
etc & $dr = \alpha(t,r')\,dt + \beta(t,r')\,dr'$

Hence ds^2 of form

$$\Rightarrow ds^2 = -\tilde{C}(t,r')\,dt^2 + \tilde{D}(t,r')\,dr'^2 + 2\tilde{E}(t,r')\,dr'\,dt + r'^2\,d\Omega^2$$

for some functions $\tilde{C}$, $\tilde{D}$ etc, which we don't need to write explicitly

Let's drop the primes & tilda's $\Rightarrow$

$$ds^2 = -C(t,r)\,dt^2 + D(t,r)\,dr^2 + 2E(t,r)\,dr\,dt + r^2\,d\Omega^2. \qquad (*)$$

2) let $dt' = \eta(t,r)\left[C(t,r)\,dt - E(r,t)\,dr\right]$

where choose η such that this is an exact differential

ie $dt' = \dfrac{\partial t'}{\partial t}dt + \dfrac{\partial t'}{\partial r}dr$

So $\frac{\partial}{\partial r}(\eta C) = \frac{\partial}{\partial t}(-E\eta)$

In principle can integrate this to find $\eta(t,r)$ given some $\eta(t_0, r)\ \forall r$

Then $dt'^2 = \eta^2\left(C^2 dt^2 - 2CE\, dt\, dr + E^2 dr^2\right]$

so $-\frac{dt'^2}{\eta^2}\frac{1}{C} = -C\, dt^2 + 2E\, dt\, dr - E^2/C\, dr^2$

$$\Rightarrow \quad -C\, dt^2 + 2E\, dt\, dr = -\frac{dt'^2}{\eta^2 C} + E^2/C\, dr^2$$

Substitute into ⊛ $\Rightarrow$

$$ds^2 = -\frac{dt'^2}{\eta^2 C} + \left(D + \frac{E^2}{C}\right) dr^2 + r^2 d\Omega^2$$

Let $B(t,r) \equiv \frac{1}{\eta^2(t,r)\, C(t,r)}$

$A(t,r) = D(t,r) + \frac{E^2(t,r)}{C(t,r)}$

& relabel t' by t $\Rightarrow$

$$\boxed{ds^2 = -B(t,r)\, dt^2 + A(t,r)\, dr^2 + r^2 d\Omega^2}$$