Université Paris Diderot (Paris 7)

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GENERAL RELATIVTY
NPAC

TD 4

1 Black hole with a cosmological constant

Consider Einstein's equations with a 'cosmological constant', that is a stress-energy tensor of the form $T_{\mu\nu} = -\Lambda g_{\mu\nu}$.

1. Use the Bianchi identity, or equivalently conservation of energy momentum, to show that Λ =constant.
2. Solve the equations (setting $8\pi G = 1$ for simplicity) in a generally static and spherically symmetric space-time to find the generalisation of the Schwarzschild black-hole.
3. What are the Killing vectors of this space-time ? What quantities are conserved along geodesics ?
4. Write down the equation for radial geodesics, in terms of an effective potential (just as we did for the Schwarzschild black-hole). Sketch the effective potential for massive particles, and discuss their motion (briefly).

2 Interior solutions again

Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional space-time, with line element $ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Psi(r)}dr^2 + r^2 d\theta^2$.

1. Derive the analogue of the Tolman-Oppenheimer-Volkov (TOV) equation, namely the equation for dp/dr where $p(r)$ is the pressure of the perfect fluid (we did the analogous thing for the interior solution with Spherical symmetry in (3+1) dimensions).
2. Show that the vacuum solution can be written as
$$ds^2 = -dt^2 + \frac{1}{1-8GM}dr^2 + r^2 d\theta^2 \tag{1}$$
where M is a constant.
3. Solve the (2+1) TOV equation for a constant density star. Find $p(r)$ and solve for the metric.
4. Solve the (2+1) TOV equation for a star with equation of state $p = \kappa\rho^{3/2}$ where κ is a constant. Find $p(r)$ and solve for the metric.

3 Linearised Einstein equations and GWs

Decompose the metric into the flat Minkowski metric, plus a small perturbation :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

with $|h_{\mu\nu}| \ll 1$. We restrict ourselves to coordinates in which $\eta_{\mu\nu}$ takes its canonical form $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. Write down the following quantities to *linear order* in the perturbation :

1. The inverse metric $g^{\mu\nu}$. If your expression contains $h^{\mu\nu}$, explain how this is obtained from $h_{\alpha\beta}$ (i.e. what metric do you use to raise the indices ?)
2. The Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$.
3. The Riemann tensor $R_{\mu\nu\rho\sigma}$.
4. The Ricci tensor $R_{\alpha\beta}$.
5. The Ricci scalar R.
6. The Einstein tensor $G_{\alpha\beta}$.
7. Does your Einstein tensor satisfy $\partial^{\mu} G_{\mu\nu} = 0$? Why *should* it satisfy this ?
8. Show that the Einstein tensor of part 6 can be obtained by varying the following Lagrangian $\mathcal{L}$ with respect to $h_{\mu\nu}$:
$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^{\mu}{}_{\sigma}) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right] \quad (2)$$
where $h = h^{\alpha}{}_{\alpha}$. Derive this Lagrangian yourself, using the result of part 5 of this exercise.
9. Write down the Einstein equations in terms of the trace-reversed perturbation defined in lectures :
$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \quad (3)$$
10. Now impose the Lorentz gauge, $\partial^{\mu}\bar{h}_{\mu\nu} = 0$. Show that *in the absence of matter*, the linearised Einstein equations become
$$\Box \bar{h}_{\mu\nu} = 0 \quad (4)$$
These equations are very similar to Maxwells equations in empty space : the only difference is that the perturbations are associated with a metric tensor (2 indices). Convince yourself that equation (4) is nothing other than the wave equation. Show that a solution for a wave travelling in the z-direction, is
$$\bar{h}_{\mu\nu} = H_{\mu\nu} e^{i k_\alpha x^\alpha} \quad (5)$$
where $H_{\mu\nu}$ is the polarisation tensor and
$$k^{\mu} = (\omega, 0, 0, \omega) \quad (6)$$
with $k^2 = 0$. What is the speed of propagation of the gravitational wave ?

4 Electromagnetism and the TT gauge [from D.Langlois book]

The aim of this exercise is to understand the TT gauge, using electromagnetism as a helpful example.

1. The electromagnetic Lagrangian $L \propto \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$ is invariant under the $U(1)$ gauge transformations $A_\mu \to A_\mu + \partial_\mu \chi$. Use this invariance to show that one can always choose the Lorentz gauge $\partial_\mu A^\mu = 0$.

2. Write down Maxwells equations (in the vacuum) in the Lorentz gauge. Show that there exists a residual gauge freedom, and use it to fix $A_0 = 0$. (Note that the solution of the wave eqaution $\partial_\mu \partial^\mu f = 0$ with initial conditions $f = 0$ and $\partial_t f = 0$ on a hypersurface of t=constant, is $f = 0$.)
3. Using the above, show that for gravitational waves propagating in empty space, one can impose the TT gauge.

5 Killing vectors [From exam 2018]

1. Explain, in a few words, what information is contained in Killing vectors. Show that in a space-time with metric $g_{\alpha\beta}$ and Riemann tensor $R^\rho{}_{\sigma\mu\nu}$, any Killing vector K^μ satisfies

$$\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu. \tag{7}$$

   [Hint : use the equation defining the Riemann tensor in terms of a commutator of covariant derivatives, together with the Killing equation, and the symmetries of the Riemann tensor]

2. **If and only if you have finished all the other exercises, and you have spare time :** Deduce from (7) that $K^\lambda \nabla_\lambda R = 0$, from which one concludes that the Ricci scalar does not change as we move along a Killing vector field.

   [Hint : a) contract ρ and μ. b) Then apply ∇^σ to both sides of the resulting expression. c) Using, amongst other things, the Killing equation, show that $\nabla^\sigma \nabla_\mu \nabla_\sigma K^\mu = 0$.]