

GENERAL RELATIVITY

NPAC

TD 3

1 Solutions of Einsteins equations

1. Compute explicitly the non-vanishing Christoffel symbols and components of the Ricci tensor for the Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \left(1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\Omega^2$$

2. Identify the change of coordinates $\bar{t} = \bar{t}(t, r)$, $\bar{r} = \bar{r}(t, r)$ which transforms the de Sitter metric :

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega^2)$$

(which we will discuss further in the section on cosmology) with H =constant, into the following form

$$ds^2 = - \left(1 - \frac{\bar{r}^2}{R_H^2} \right) d\bar{t}^2 + \left(1 - \frac{\bar{r}^2}{R_H^2} \right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

where $R_H = H^{-1}$. The first metric is known as the flat form of the de Sitter metric, and the second one as the static form of the de Sitter metric.

[Hint : $e^{Ht} = e^{H\bar{t}} \sqrt{1 - H^2 \bar{r}^2}$ and $re^{Ht} = \bar{r}$.]

2 [Exam 2018] Conformally flat metrics

In two dimensions (time together with one spatial direction), a general line-element can be written locally as

$$ds^2 = \Omega^2(t, x)(-dt^2 + dx^2)$$

where $\Omega(t, x)$ is an arbitrary non-vanishing function of t and x . The factor $\Omega(t, x)$ which multiplies the Minkowski metric, is known as the *conformal factor*, and the above metric is said to be *conformally flat*.

- i) Write down the Lagrangian from which one determines the equation of motion for time-like geodesics. Together with the geodesic equation, use it to determine the Christoffel symbols for this metric.
- ii) Verify your calculations by determining the Christoffel symbols by direct calculation from the metric.
- iii) Using the symmetries of the Riemann tensor, determine the number N of independent components of the Riemann tensor in 2 dimensions. Calculate these N components.

3 Schwarzschild metric in different coordinate systems

In some problems it is useful to use alternative, non-standard, coordinates for the Schwarzschild metric. Here are two examples.

1. Isotropic coordinates

Let X, Y, Z be new coordinates related to r, θ, ϕ (and hence $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$) by

$$r = R \left(1 + \frac{m}{2R} \right)$$

and

$$X = Rr^{-1}x, \quad Y = Rr^{-1}y, \quad Z = Rr^{-1}z$$

In terms of these coordinates, show that the Schwarzschild metric becomes

$$ds^2 = - \left(\frac{2R - m}{2R + m} \right)^2 dt^2 + \left(1 + \frac{m}{2R} \right)^4 (dX^2 + dY^2 + dZ^2)$$

Hence you can show that spaces with $t=\text{constant}$ in the Schwarzschild metric are conformal to Euclidean space.

2. Regge-Wheeler/tortoise coordinates

In the region $r > 2m$, define the tortoise radial coordinate by

$$\rho = r + 2m \log(r - 2m)$$

Write down the Schwarzschild metric in (t, ρ, θ, ϕ) coordinates. Show that it takes a form in which the time-like sections $\theta = \text{constant}, \phi = \text{constant}$ are conformal to 2-dimensional Minkowski space.

4 Spherically symmetric metrics

Spherical symmetry implies that any line element can be written in the form

$$ds^2 = -C(t, r)dt^2 + D(t, r)dr^2 + 2E(t, r)drdt + F(t, r)r^2d\Omega^2.$$

To write this in standard form, make the following change of coordinates :

1. Go from (t, r) to (t, r') coordinates where $r'^2 = F(t, r)r^2$ (and invertibility is assumed)
2. Now label r' as r again. Show that the cross term can be removed by setting $dt'^2 = \eta(t, r)[C(t, r)dt - E(t, r)dr]$
3. Finally, show that the resulting metric takes the form (on labelling t' as t again)

$$ds^2 = -B(t, r)dt^2 + A(t, r)dr^2 + r^2d\Omega^2$$

and find the link between A, B and the other variables. This is the most useful form in which to write a general spherically symmetric metric.

5 Shapiro Effect [From D.Langlois book]

The aim is to calculate the time taken by a light-ray (or radar signal) to go from the earth (E) to a planet (P) and back again.

1. Consider a light signal propagating in the Schwarzschild metric. Denote by r_0 the minimum radius of the trajectory of the light ray. Determine dt/dr as a function of r as well as the constants M , r_0 , and the conserved angular momentum L and energy E .
2. Now expand the RHS of your expression in a perturbative expansion in powers of m/r (i.e. assume $m \ll r$). Use this to calculate the time taken to go from the earth to the planet and back again, as a function of r_0 , r_E and r_P (where this last two are the radial coordinates of the earth and the planet).

6 Kerr metric and Killing vectors [From D.Langlois book]

The Kerr metric is given by

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mar \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2 \quad (1)$$

where $0 \leq a \leq M$ and

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2 \\ \rho^2 &= r^2 + a^2 \cos^2 \theta \end{aligned}$$

1. What can you say about the $a \rightarrow 0$ limit?
2. Consider circular trajectories in the equatorial plane of the Kerr BH ($\theta = \pi/2$).
 - (a) Show that the orbital period measured by an asymptotic observer is

$$T_\infty(r) = \frac{2\pi}{1 - \frac{2M}{r}} \left(\sqrt{\Delta} - \frac{2Ma}{r} \right),$$

for a trajectory that is in *corotation* with the blackhole ($\dot{\phi} > 0$). Also find the orbital period for a trajectory in *anti-rotation*.

- (b) Now focus on the case of corotation. Show that the radial coordinate r_{\min} corresponding to the minimal period satisfies the equation

$$r(r - 2M)(r - 3M) - 2Ma^2 + 2Ma\sqrt{\Delta} = 0 \quad (2)$$

3. Now consider *lightlike geodesics* in the plane $\theta = \pi/2$.
 - (a) Identify the two Killing vectors of the metric (1).
 - (b) From them, determine the two conserved quantities.
 - (c) Deduce that the geodesics satisfy

$$\dot{r}^2 = \alpha^2 \left[1 + \frac{a^2 - b^2}{r^2} + \frac{2M(a - b)^2}{r^3} \right]$$

where α and b need to be expressed in terms of the two constants of motion.

- (d) For a circular geodesic, determine b as a function of the radius r_{circ} of the geodesic. Show that r_{circ} is a solution of (2).

7 [Exam 2018] Another coordinate transformation

1. The space-time geometry around a static spherically symmetric object of mass M can be described by a line-element of the form

$$ds^2 = -F(r)dt^2 + H(r)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2). \quad (3)$$

(Notice that the function $H(r)$ also multiplies the part in $r^2d\Omega^2$.) In the limit of a weak gravitational field, an approximate expression for the functions F and H is the following :

$$F(r) = 1 - 2\frac{GM}{r} + 2\beta\left(\frac{2GM}{r}\right)^2 + \dots \quad (4)$$

$$H(r) = 1 + 2\gamma\frac{GM}{r} + \dots, \quad (5)$$

where β and γ are parameters (constants) called ‘post-newtonian parameters’.

- i) Explain in a few words the origin of the first non-trivial term in $F(r)$. Will these approximate expressions for F and H be a good description for the metric around a neutron star? And for the sun?
- ii) Using a known static spherically symmetric metric of your choice, and after carrying out the necessary coordinate transformations, determine the values of β and γ in General Relativity.

[In modified gravity theories, different values of β and γ are allowed.]

8 [Exam 2018] Geodesics in modified gravity

For this exercise, we work with the metric (3) where the functions $F(r)$ and $G(r)$ are given in (4) and (5) respectively.

1. Consider two observers at fixed positions : the first is at $r = r_1$, and the second is at $r = r_2 > r_1$. The first observer sends two successive light signals to the second observer. The proper time interval between the two light signals is $\delta\tau_1$ for the first observer, and $\delta\tau_2$ for the second observer.
 - The first light signal is emitted at coordinate time t_1 and arrives at r_2 at coordinate time t_2 . The second light signal is emitted at coordinate time $t_1 + \delta t_1$ and arrives at r_2 at coordinate time $t_2 + \delta t_2$. Show that $\delta t_1 = \delta t_2$.
 - The frequency ν of the electromagnetic signals is related to the proper time interval by $\nu = 1/\delta\tau$. Show that the ratio ν_2/ν_1 (where ν_2 =received frequency, and ν_1 =emitted frequency) is given by $\sqrt{F(r_1)/F(r_2)}$. Would this frequency shift be visible on earth (say with the two observers separated by a few hundred meters)?
2. Identify all the Killing vectors of the metric (3). Write down the conserved quantities along geodesics describing the dynamics of free particles, and give them a physical interpretation.
3. Study light-like geodesics in the space-time (3), in the plane $\theta = \pi/2$. Find the first order equation for $r(\phi)$. In terms of the variable $u = b/r$, where the constant b should be determined, show that this equation takes the form

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{H}{F}. \quad (6)$$

4. Working to first order in GM/r , write the explicit solution of (6) in terms of the post-newtonian parameter γ . [The solution of $(du/d\phi)^2 + u^2 = 1 + 2\alpha u$, for some constant α , is $u = \sqrt{1 + \alpha^2} \sin(\phi - \bar{\phi}) + \alpha$.]
5. Hence deduce the angle $\Delta\phi$ through which a light ray is deflected.

9 [Exam 2019] Stress energy tensor and Einstein equations

1. The action for electromagnetism is given by

$$S_m = -\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$$

where the Maxwell field $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Show that the stress energy tensor associated with the electromagnetic field is given by

$$T_{\alpha\beta} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\alpha\beta}} = F_{\sigma\alpha} F^\sigma{}_\beta - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu}.$$

[Hint : you may use (though bonus for proving it) the result derived in TD, namely that $\delta g = g g^{\alpha\beta} \delta g_{\alpha\beta}$.]

2. Show that in the presence of an electromagnetic field, Einstein's equations can be written as

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

3. For a pure electric field $F^{0i} = E^i$, $F^{ij} = 0$, write the different components of $T^\mu{}_\nu$ (you may answer this question either using general arguments or by calculation).