M2 NPAC, Nuclear Physics: exercises #1

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Institut de Physique Nucléaire, Orsay — November 8, 2018

Introduction

This first exercise sheet contains mostly very simple exercises designed to help you familiarize yourself with the concepts and formulas seen during lessons 1 to 3. If you can't answer the questions easily it must be that something is missing in your notes. However some questions in Section 3 are more tricky.

1 Mass excess and binding energy

We got two equivalent expressions for atomic mass:

$$\mathrm{M}(A,Z)c^2 = (Z\mathrm{M_H} + N\mathrm{M_n})c^2 - \mathrm{B}(A,Z). \tag{1}$$

where $\mathrm{M_H}$ is the mass of the hydrogen atom and $\mathrm{M_n}$ the mass of the neutron, and

$$\mathrm{M}(A,Z)c^2 = A\mathrm{m_u}c^2 + \Delta(A,Z)c^2 \tag{2}$$

where $\mathrm{m_u}$ is the atomic mass unit.

Question 1

(a) recall the definitions of $\mathrm{B}(A,Z)$ and $\Delta(A,Z)$.

(b) Show the following relationship between $\mathrm{B}(A,Z)$ and $\Delta(A,Z)$ (given during the lesson but not demonstrated):

$$\mathrm{B}(A,Z) = Z\Delta(\mathrm{H})c^2 + N\Delta(\mathrm{n})c^2 - \Delta(A,Z)c^2 \tag{3}$$

(c) One has the following experimental values: $\Delta(^{40}_{20}\mathrm{Ca})c^2 = -38.847$ MeV and $\Delta(^{40}_{22}\mathrm{Ti})c^2 = -9.064$ MeV, which of this two $A = 40$ isobars is the most bound ?

Solution:

(a) $B(A,Z)$ is the binding energy of the nucleus with A, Z mass and charge numbers; $\Delta(A,Z)$ the mass excess.

(b)

$$
\begin{aligned}
B(A,Z) &= (ZM_H + NM_n)c^2 - M(A,Z) \\
&= (ZM_H + NM_n)c^2 - Am_u c^2 + \underbrace{Am_u c^2 - M(A,Z)c^2}_{\text{definition of } \Delta} \\
&= (ZM_H + NM_n)c^2 - (Zm_u + Nm_u)c^2 - \Delta(A,Z)c^2
\end{aligned}
$$

since:

$$
\begin{aligned}
M_H c^2 - m_u c^2 &\equiv \Delta(H)c^2 \\
M_n c^2 - m_u c^2 &\equiv \Delta(n)c^2
\end{aligned}
$$

it comes Eq. 3.

(c)

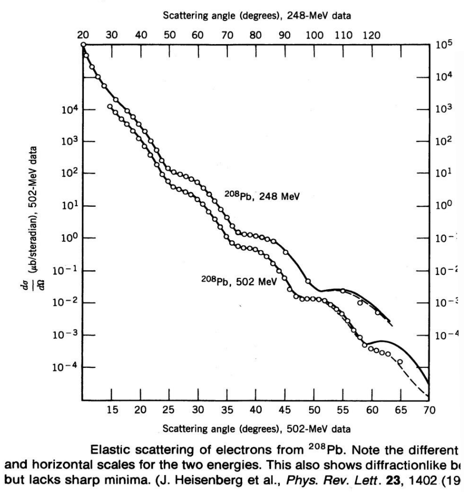
$$
\begin{aligned}
M({}^{40}_{20}\mathrm{Ca})c^2 &= 40 \cdot m_u c^2 - 38.847 \text{ [MeV]} \\
M({}^{40}_{22}\mathrm{Ti})c^2 &= 40 \cdot m_u c^2 - 9.064 \text{ [MeV]}
\end{aligned}
$$

hence $M({}^{40}_{20}\mathrm{Ca}) < M({}^{40}_{22}\mathrm{Ti}) \implies {}^{40}_{20}\mathrm{Ca}$ more bound than ${}^{40}_{22}\mathrm{Ti}$.

2 Radius from electron scattering

The momentum transfer to an electron of energy E_e scattered at θ angle (with respect to the beam axis) in an electron-nucleus elastic collision is:

$$
|\vec{q}| = q = 2k \sin\left(\frac{\theta}{2}\right) = 2\frac{p}{\hbar} \sin\left(\frac{\theta}{2}\right) = \frac{2}{\lambda} \sin\left(\frac{\theta}{2}\right) \qquad (4)
$$



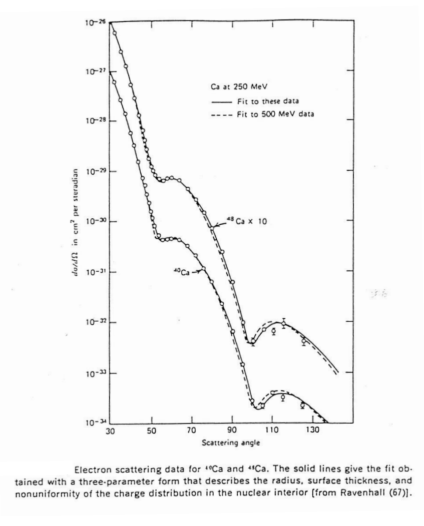


Fig. 1

Question 2

Use Fig. 1 to obtain a rough estimate of the charge radii of Pb and Ca.
Hint. A simple estimate of these radii can be obtained by observing that the successive minima or maxima in the diffraction patterns are separated by angles corresponding to $\Delta(qR) \simeq \pi$. Use $\hbar c = 197$ MeV · fm.

Solution: First we remind that $\lambda = \frac{2\pi\cdot 197}{E_e}$ or $\lambda\!\!\!^{-} = \frac{\lambda}{2\pi} = \frac{197}{E_e}$.
$\Delta(qR) = \pi$ i.e. $R\Delta q = \pi$ and from Eq. 4 $\Delta(q) = \frac{2}{\lambda\!\!\!^{-}}\Delta\left[\sin\left(\frac{\theta}{2}\right)\right]$. Hence

$$R = \frac{\lambda\!\!\!^{-}\pi}{2\Delta\left[\sin\left(\frac{\theta}{2}\right)\right]} = \frac{\lambda}{4\Delta\left[\sin\left(\frac{\theta}{2}\right)\right]}.$$

For the ^{208}Pb case at $E_e = 502$ MeV, $\lambda = 2.466$ fm or $\lambda\!\!\!^{-} = 0.392$ fm. The first minimum occurs around 25°, the second around 35° $\implies \Delta\left[\sin\left(\frac{\theta}{2}\right)\right] = 0.085$,

$$R = \frac{2.466}{4\cdot 0.085} = 7.25 \text{ fm}.$$

Similarly for ^{40}Ca one would find $R \simeq 3.7$ fm.

3 Symmetry energy

Notice: This section is slightly more seriously difficult. The following set of exercises could have typically been part of an exam subject.

We will treat in this section mass 16.
Lesson reminders. We call n_+ and n_- the numbers of symmetric and anti-symmetric pairs of nucleons inside the nucleus, respectively. p_+ the probability to find a symmetric pair within interaction range b and p_- the same probability but for anti-symmetric pairs. We will assume that $b \simeq 1.7$ fm. For a nuclear interaction with exchange character the average nuclear potential energy $\bar{V}$ is simply given by:

$$\bar{V} = -(n_+p_+ - n_-p_-)V_0 \tag{5}$$

where V_0 is the binding potential energy (defined positive) per interacting pair. We can also write Eq. 5 as:

$$\bar{V} = -p(n_+g_+ - n_-g_-)V_0 \tag{6}$$

where p is the total number of pairs within range b and g_+ (g_-) is the relative probability to find a symmetric (anti-symmetric) pair of nucleons within the nuclear interaction range b.
We also assume that each nucleon occupies an average sphere volume of radius d.

Question 3

(a) Consider a nucleus of mass number A, express p as a function of b, d and A.

(b) Use materials in slide set n°1 to find an expression for $\bar{V}$ as a function of n_+, n_-, A and V_0 (explain which figures you used in slide set n°1).

Solution:

(a)

$$p = \frac{\frac{4}{3}\pi b^3}{\frac{4}{3}\pi R^3}$$

where R is the nuclear radius, $R = d \cdot A^{1/3}$, hence

$$p = \frac{b^3}{d^3} \times \frac{1}{A}$$

(b) With the left-hand figure in slide n°9 (R_{rms} as a function of $A^{1/3}$) we see that $d = 1.23$ fm $\Longrightarrow$ $d/b = 1.23/1.7 = 0.72$. If we report this value in left-hand figure in slide n°14 we see that $g_+ \simeq 1.75$ and $g_- \simeq 0.25$. It comes:

$$\bar{V} = -\frac{b^3}{d^3} \times \frac{1}{A}(1.75n_+ - 0.25n_-)V_0 = -\frac{1.7^3}{1.23^3} \times \frac{1}{A}(1.75n_+ - 0.25n_-)V_0 = -\frac{2.64}{A}(1.75n_+ - 0.25n_-)V_0$$

We can now try to determine V_0 from the mass data of the $A = 16$ isobaric chain. The measured mass excess of the $A = 16$ isobaric chain members are:

Nucleus:	${}^{16}_{10}Ne_6$	${}^{16}_{9}F_7$	${}^{16}_{8}O_8$	${}^{16}_{7}N_9$	${}^{16}_{6}C_{10}$	${}^{16}_{5}B_{11}$
$\Delta({}^{16}X)c^2$ [MeV]=	+23.986	+10.680	−4.737	+5.684	+13.694	+37.112

Question 4

(a) Draw the nucleon $4n$ normal-arrangement partitions for the $A = 16$ isobar chain from $Z = 5$ to $Z = 10$ (both included), name the nuclei, give the corresponding T_3 isospin projection number.

(b) Plot $\Delta({}^{16}X)c^2$ (MeV) as a function of T_3. Calculate the charge energy difference between all isobars and ${}^{16}O$ (i. e. use ${}^{16}O$ as a reference point) using $M_n c^2 - M_p c^2 = 1.29$ MeV, the value of the fine structure constant $e^2/\hbar c = 1/137$ and $\hbar c = 197$ MeV · fm and assuming that the charge of the nucleus (Ze) is homogeneously distributed in a sphere of radius $R = d \cdot A^{1/3}$ with $d = 1.23$ fm (as used previously). Correct the experimental mass excess from charge effect, display the result on the same drawing.

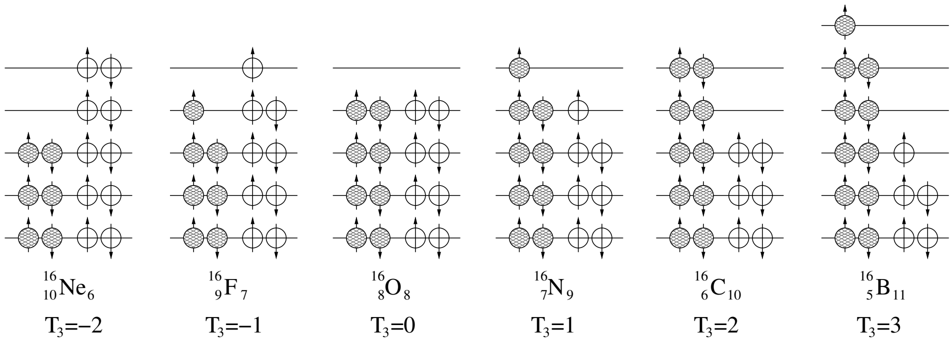
(c) Once the experimental mass excess has been corrected from charge effect only remains the pure nuclear bind effect as given by $\bar{V}$. Evaluate graphically $n_+ - n_-$ from your partition drawing. Noting that $n_+ + n_- = 1/2 \times A(A-1)$ deduce n_+ and n_- for ${}^{16}F$, ${}^{16}O$ and ${}^{16}N$. Express $\bar{V}$ for ${}^{16}N$, ${}^{16}O$ and ${}^{16}F$ as a function of V_0.

(d) What should be the difference $\bar{V}({}^{16}N) - \bar{V}({}^{16}F)$ (if you could not answer to the previous question use a qualitative argument)? Is it consistent with the result you obtained at question (b) above? If not, try and identify from where comes the problem.

(e) From the values of the mass excess *corrected for charge effect* that you obtained in (b) for ${}^{16}O$ and ${}^{16}N$ deduce V_0.

Solution:

(a) Partitions:

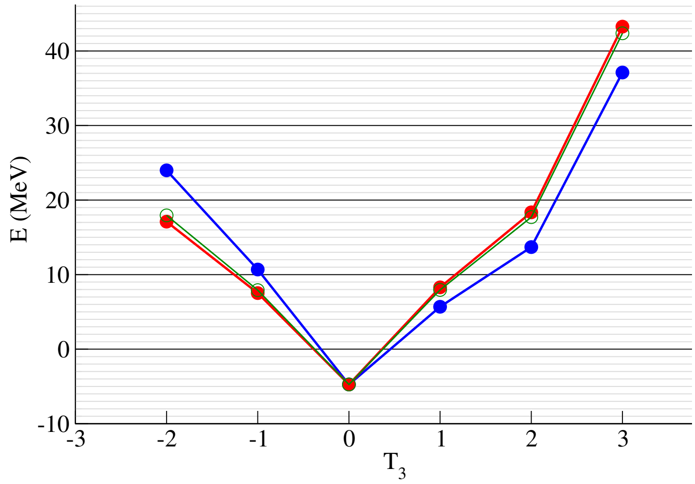


(b) Since the graph is plotted as a function of T_3 for fixed A, it is convenient to use the expression of the total (including both Coulomb and proton-neutron mass difference) contribution of the charge effect to the binding energy as given during Lesson 2:

$$\Delta E_c = E_c(T_3) - E_c(T_3') = (T_3 - T_3') \times \left[(M_n c^2 - M_p c^2) - \frac{6}{10}(A - 1 - T_3 - T_3')\frac{e^2}{R} \right]$$

We consider ^{16}O as the reference, $T_3' = 0$. T_3 takes the values −2 to +3, and we obtain:

Nucleus:	$^{16}_{10}$Ne$_6$	$^{16}_{9}$F$_7$	$^{16}_{8}$O$_8$	$^{16}_{7}$N$_9$	$^{16}_{6}$C$_{10}$	$^{16}_{5}$B$_{11}$
T_3 =	−2	−1	0	+1	+2	+3
ΔE_c [MeV]=	+6.844	+3.164	0	−2.607	−4.658	−6.151
$\Delta(^{16}X)c^2 - \Delta E_c$ =	+17.101	+7.516	−4.737	+8.291	+18.351	+43.263



Blue: experimental mass excess values. Red: corrected for charge effect. (Green open circles: see answer (d) below).

(c) For ^{16}O:

$$n_+ - n_- = 6 \times \binom{16}{4} - 4 \times \frac{1}{2}\frac{16}{4}\left(\frac{16}{4} - 1\right) = 0$$

$$2n_+ = (n_+ - n_-) + (n_+ + n_-) = 0 + \frac{1}{2}16(16-1) = 120$$

$$\Longrightarrow n_+ = 60$$

$$2n_- = (n_+ + n_-) - (n_+ - n_-) = 120 - 0$$

$$\Longrightarrow n_- = 60$$

(c) (continued) For ^{16}F and ^{16}N:

$$n_+ - n_- = 6 \times \binom{12}{4} + 3 + 0 - \left\{\frac{1}{2}\left[(5\times 4) + 2\times(4\times 3) + 3\times 2\right]\right\} = -4$$

$$\begin{aligned} 2n_+ &= (n_+ - n_-) + (n_+ + n_-) = -4 + \frac{1}{2}16(16-1) = 116 \\ \Longrightarrow n_+ &= 58 \\ 2n_- &= (n_+ + n_-) - (n_+ - n_-) = 120 - (-4) = 124 \\ \Longrightarrow n_- &= 62 \end{aligned}$$

We can then write:

$$\begin{aligned} \bar{V}(^{16}\mathrm{O}) &= -\frac{2.64}{16}(1.75\times 60 - 0.25\times 60)V_0 = -14.851\times V_0 \\ \bar{V}(^{16}\mathrm{N}) &= -\frac{2.64}{16}(1.75\times 58 - 0.25\times 62)V_0 = -14.191\times V_0 \\ &= \bar{V}(^{16}\mathrm{F}) \end{aligned}$$

(d) Of course $\bar{V}(^{16}\mathrm{N}) = \bar{V}(^{16}\mathrm{F})$, these are mirror nuclei, and their partitions are exactly identical (they have same numbers of symmetric and anti-symmetric pairs): they are identical from the point of view of the symmetry energy. It seems that we have "over corrected" for the charge effect (corrected binding energy values are pushed slightly too high by the charge correction for $T_3 > 0$ and too low for $T_3 < 0$). A way to decrease the charge correction is to slightly increase the nuclear (charge) radius. We have assumed $R = d \cdot A^{1/3}$ with $d = 1.23$ fm, this d values actually is a r_{0rms} stemming from a poor fit of the curve in the low mass region [see left-hand figure in slide n°9 (R_{rms} as a function of $A^{1/3}$)]: it is naturally tempting to increase d (aka r_0). A value of $d = 1.35$ gives an almost perfect agreement as shown by the green curve in the mass-excess figure above.

(e)

$$\begin{aligned} [\Delta(^{16}\mathrm{N})c^2 - \Delta E_c((^{16}\mathrm{N})] - \Delta(^{16}\mathrm{O})c^2 &= 8.291 + 4.737 = 13.028 \\ &= \bar{V}(^{16}\mathrm{N}) - \bar{V}(^{16}\mathrm{O}) = (-14.191 + 14.851)V_0 = 0.660V_0 \\ \Longrightarrow &\boxed{V_0 = 19.7 \text{ MeV}} \end{aligned}$$

(quite a reasonable estimate, V_0 is known to be $\simeq 21$ MeV in the O region).

In the mass evaluation tables we find the following mass excess measured values for ^{15}O, ^{15}F and ^{1}H:

nucleus	^{15}N	^{15}O	^{15}F	^{1}H
Δc^2 [MeV] =	0.101	2.855	16.810	7.289

Question 5

(a) Calculate the proton separation energy for ^{16}O, ^{16}F and ^{16}Ne.

(b) Where is located the proton drip line of the $A = 16$ chain?

(c) Predict the resonance energy of the unbound ^{16}F. The actual experimental value is +50 keV. What could explain the difference ?

Solution:

(a)

$$
\begin{aligned}
S_p(^{16}\mathrm{O}) =& \mathrm{M}(^{15}\mathrm{N})c^2 + \mathrm{M}(^{1}\mathrm{H})c^2 - \mathrm{M}(^{16}\mathrm{O})c^2 \\
S_p(^{16}\mathrm{F}) =& \mathrm{M}(^{15}\mathrm{O})c^2 + \mathrm{M}(^{1}\mathrm{H})c^2 - \mathrm{M}(^{16}\mathrm{F})c^2 \\
S_p(^{16}\mathrm{Ne}) =& \mathrm{M}(^{15}\mathrm{F})c^2 + \mathrm{M}(^{1}\mathrm{H})c^2 - \mathrm{M}(^{16}\mathrm{Ne})c^2 \\
\Longrightarrow S_p(^{16}\mathrm{O}) &= +12.127\ \mathrm{MeV} \\
S_p(^{16}\mathrm{F}) &= -0.536\ \mathrm{MeV} \\
S_p(^{16}\mathrm{Ne}) &= +0.113\ \mathrm{MeV}
\end{aligned}
$$

(b) This is a strange situation because S_p which is obviously positive for the stable (tightly bound) ^{16}O, becomes negative already for ^{16}F. Strictly speaking the proton drip line is defined by $S_p = 0$ therefore the drip line is located between ^{16}O and ^{16}F. However S_p becomes slightly positive again with ^{16}Ne! Anyway, experimentally both ^{16}F and ^{16}Ne are observed as proton unbound resonances and the proton drip line is indeed just... next to the stable ^{16}O.

(c) The proton in ^{16}F is unbound by $-S_p = 0.536$ MeV. This is at this excitation energy that a resonance could be found. The only argument you could make at this stage of the course is that ^{16}F is an odd-odd nucleus and that additional binding energy may arise from a spin aligned proton-neutron configuration (as for deuterium).