SOLUTIONS TO EXERCISES OF 16/12/19

Exercise 1

We know that $\vec{T} = \frac{\vec{\tau}_1}{2} + \frac{\vec{\tau}_2}{2}$ and $\vec{\tau}^2 = 3$.

Then $4\vec{T}^2 = \vec{\tau}_1^2 + \vec{\tau}_2^2 + 2\vec{\tau}_1 \cdot \vec{\tau}_2 \Rightarrow \vec{\tau}_1 \cdot \vec{\tau}_2 = 2\vec{T}^2 - 3$

$$\Rightarrow (\tau_1 \cdot \tau_2)|\alpha T M_T\rangle = \left(2\vec{T}^2 - 3\right)|\alpha T M_T\rangle$$

$$= \left[2T(T+1) - 3\right]|\alpha T M_T\rangle$$

$$= (\underbrace{2T^2 + 2T}_{=4T \text{ because } T^2 = T \text{ for } T=0,1} - 3)|\alpha T M_T\rangle$$

$$= (4T - 3)|\alpha T M_T\rangle \quad (*)$$

$$\Rightarrow (\tau_1 \cdot \tau_2)^2 |\alpha T M_T\rangle = (16T^2 - 24T + 9)|\alpha T M_T\rangle$$

$$= (9 - 8T)|\alpha T M_T\rangle$$

$$= \left[3 - 2(4T-3)\right]|\alpha T M_T\rangle$$

$$= (3 - 2\vec{\tau}_1 \cdot \vec{\tau}_2)|\alpha T M_T\rangle$$

where $(*)$ has been used

Exercise 2

$$\langle \alpha\beta | \alpha\beta \rangle = \langle 0 | a_\beta a_\alpha a_\alpha^\dagger a_\beta^\dagger | 0 \rangle$$

$$= \langle 0 | a_\beta \left(1 - a_\alpha^\dagger a_\alpha\right) a_\beta^\dagger | 0 \rangle$$

$$= \langle 0 | a_\beta a_\beta^\dagger | 0 \rangle - \langle 0 | a_\beta a_\alpha^\dagger a_\alpha a_\beta^\dagger | 0 \rangle$$

$$= \langle 0 | (1 - a_\beta^\dagger a_\beta) | 0 \rangle - \langle 0 | a_\beta a_\alpha^\dagger (- a_\beta^\dagger a_\alpha | 0 \rangle$$

$$= \langle 0 | 0 \rangle - \underbrace{\langle 0 | a_\beta^\dagger a_\beta | 0 \rangle}_{= 0} + \underbrace{\langle 0 | a_\beta a_\alpha^\dagger a_\beta^\dagger a_\alpha | 0 \rangle}_{= 0}$$

$$= 1$$

Exercise 3

In first quantisation

$$O^{(5)} = \frac{1}{5!} \sum_{i \neq j \neq k \neq m \neq n = 1}^{N} O(i, j, k, m, n)$$

In second quantisation

$$O^{(5)} = \frac{1}{5!} \sum_{\alpha\beta\gamma\delta\mu\nu\varepsilon\xi\rho\lambda} O_{\alpha\beta\gamma\delta\mu\nu\varepsilon\xi\rho\lambda} \, a_\alpha^\dagger a_\beta^\dagger a_\gamma^\dagger a_\delta^\dagger a_\mu^\dagger a_\lambda a_\rho a_\xi a_\varepsilon a_\nu$$

Exercise 4

$$|\alpha\beta\gamma\rangle = a^\dagger_\alpha a^\dagger_\beta a^\dagger_\gamma |0\rangle$$

$$N = \sum_\mu a^\dagger_\mu a_\mu$$

Then

$$N|\alpha\beta\gamma\rangle = \sum_\mu a^\dagger_\mu a_\mu a^\dagger_\alpha a^\dagger_\beta a^\dagger_\gamma |0\rangle$$

$$= \sum_\mu a^\dagger_\mu \left(\delta_{\mu\alpha} - a^\dagger_\alpha a_\mu\right) a^\dagger_\beta a^\dagger_\gamma |0\rangle$$

$$= a^\dagger_\alpha a^\dagger_\beta a^\dagger_\gamma |0\rangle - \sum_\mu a^\dagger_\mu a^\dagger_\alpha \left(\delta_{\beta\mu} - a^\dagger_\beta a_\mu\right) a^\dagger_\gamma |0\rangle$$

$$= |\alpha\beta\gamma\rangle - a^\dagger_\beta a^\dagger_\alpha a^\dagger_\gamma |0\rangle + \sum_\mu a^\dagger_\mu a^\dagger_\alpha a^\dagger_\beta \left(\delta_{\mu\gamma} - a^\dagger_\gamma a_\mu\right)|0\rangle$$

$$= |\alpha\beta\gamma\rangle + a^\dagger_\alpha a^\dagger_\beta a^\dagger_\gamma |0\rangle + a^\dagger_\gamma a^\dagger_\alpha a^\dagger_\beta |0\rangle + 0$$

$$= 3|\alpha\beta\gamma\rangle$$

Exercise 5

Wick's theorem w.r.t. particle vacuum $|0\rangle$.

a) $$\langle 0| a^\dagger_\alpha a_\beta a^\dagger_\gamma a_\delta a^\dagger_\lambda a_\mu |0\rangle = \langle 0| : a^\dagger_\alpha a_\beta a^\dagger_\gamma a_\delta a^\dagger_\lambda a_\mu : |0\rangle$$

$$+ \langle 0| : a^\dagger_\alpha a_\beta a^\dagger_\gamma a_\delta a^\dagger_\lambda a_\mu : |0\rangle$$

$$+ \langle 0| : a^\dagger_\alpha a_\beta a^\dagger_\gamma a_\delta a^\dagger_\lambda a_\mu : |0\rangle$$

$$+ \langle 0| : a^\dagger_\alpha a_\beta a^\dagger_\gamma a_\delta a^\dagger_\lambda a_\mu : |0\rangle$$

$$+ \langle 0| : a^\dagger_\alpha a_\beta a^\dagger_\gamma a_\delta a^\dagger_\lambda a_\mu : |0\rangle$$

$$= 0$$

All other contractions vanish (see page 4 of chapter 4 of the notes).

Since no fully-contracted terms survive, the expectation value is 0.

b) $\langle 0| a_\alpha a^+_\beta a_\gamma a^+_\delta a_\lambda a^+_\mu |0\rangle = \langle 0| : a_\alpha a^+_\beta a_\gamma a^+_\delta a_\lambda a^+_\mu : |0\rangle$

$+ \langle 0| : a_\alpha a^+_\beta a_\gamma a^+_\delta a_\lambda a^+_\mu : |0\rangle$

$+$ all others with one contraction $a a^+$

$+ \langle 0| : a_\alpha a^+_\beta a_\gamma a^+_\delta a_\lambda a^+_\mu : |0\rangle$

$+$ all others with two contractions $a a^+ \; a a^+$

$+ \langle 0| : a_\alpha a^+_\beta a_\gamma a^+_\delta a_\lambda a^+_\mu : |0\rangle$

only the fully contracted term survives

$= \delta_{\alpha\beta}\,\delta_{\gamma\delta}\,\delta_{\lambda\mu}$

c) $\langle 0| a^+_\alpha a^+_\beta a^+_\gamma a^+_\delta |0\rangle = 0$

Exercise 6

Express $|\phi\rangle = \sum_k c_k |\Psi_k\rangle$ with $H|\Psi_k\rangle = E_k |\Psi_k\rangle$

$$\frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{\sum_{kk'} c_k c_{k'} \langle\Psi_k|H|\Psi_{k'}\rangle}{\sum_{mm'} c_m c_{m'} \langle\Psi_m|\Psi_{m'}\rangle} = \frac{\sum_k c_k^2 E_k}{\sum_m c_m^2} \geq E_0 + \frac{\sum_{k\neq 0} c_k^2 E_k}{\sum_m c_m^2} \geq E_0$$