Exercises (4)

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8 Energetics of supernova remnants

We have seen that the evolution of a supernova remnant shock goes through 4 distinct phases: free-expansion, adiabatic, radiative (snowplough), and momentum conserving. For the expansion in an homogeneous and cold gas, the scaling of the shock radius with time can be described by a scale free function $R_s \propto t^\delta$, where $\delta = 1, 2/5, 2/7, 1/4$ for each of the four phases.

Let's call E_{SN} the explosion energy and $E(t)$ the total energy in the system at a given time.

Plot the behaviour of $E(t)$.

What is the ratio between the kinetic energy E_k and $E(t)$ in each of the phases?

9 Jump conditions at weak shocks

The conservation of mass, momentum, and energy across a shock surface are described by the following equations:

$$\varrho_1 u_1 = \varrho_2 u_2 \quad (1)$$

$$\varrho_1 u_1^2 + P_1 = \varrho_2 u_2^2 + P_2 \quad (2)$$

$$\frac{1}{2}\varrho_1 u_1^3 + \frac{5}{2}P_1 u_1 = \frac{1}{2}\varrho_2 u_2^3 + \frac{5}{2}P_2 u_2 \quad (3)$$

where ϱ_i, u_i, and P_i represent the density, velocity, and pressure of the fluid upstream ($i = 1$) and downstream ($i = 2$) of the shock. We assumed that the fluid is made of a monoatomic gas ($\gamma = 5/3$) and that its equation of state is adiabatic ($P = K\varrho^{5/3}$). The compression factor of the shock is $r = u_1/u_2 = \varrho_2/\varrho_1$. The shock Mach number is $\mathcal{M}^2 = 3(\varrho_1 u_1^2)/(5P_1)$.

1. Show that the compression factor depends on the Mach number as:

$$r = \frac{4\mathcal{M}^2}{\mathcal{M}^2 + 3} \quad (4)$$

(Hint: divide the momentum equation by $\varrho_1 u_1^2$ and the energy equation by $\varrho_1 u_1^3/2$ and combine them).

2. Shock that the jump in pressure is given by:

$$\frac{P_2}{P_1} = \frac{5\mathcal{M}^2 - 1}{4} \tag{5}$$

(Hint: use the momentum equation together with the result of point 1 above).

3. Show that the jump in temperature is given by:

$$\frac{T_2}{T_1} = \frac{(5\mathcal{M}^2 - 1)(\mathcal{M}^2 + 3)}{16\mathcal{M}^2} \tag{6}$$

4. Show that the Mach number of the downstream fluid is always smaller than 1.

5. What happens for $\mathcal{M} \longrightarrow \infty$?

10 The need for dark matter on super galactic scales: the Coma cluster

The best studied cluster of galaxies is the Coma cluster, located at a distance of about 100 Mpc. This corresponds to a redshift of z = 0.0231, or a recession velocity of ~ 6900 km/s. It contains ≈ 1000 galaxies of different sizes and masses in a roughly spherical region of radius ≈ 1 Mpc. From its roughly spherical shape (and from other observables) we can infer that it is a gravitationally bound system in a state of dynamical equilibrium. The velocities of individual galaxies along the line of sight are measured from the doppler shift of spectral lines. They show a dispersion of ≈ 1000 km/s around the mean value of ~ 6900 km/s. Finally, the mass *in form of stars* in the Coma cluster has been estimated from its total optical luminosity, and resulted equal to $\approx 10^{13} M_\odot$.

1. Assuming that stars are the main contributors to the mass of the Coma cluster, compute the escape velocity from the system. Compare it with the velocity dispersion and comment the result. Can you obtain an estimate of the mass of Coma from the measured velocity dispersion?

2. The Coma cluster is also a powerful X-ray sources. The X-ray emission is not associated with individual galaxies, but is rather diffuse throughout the entire volume of the cluster. The luminosity in the X-ray band is of the order of $\approx 10^{45}$ erg/s. The spectral shape suggests that the emission is due to thermal Bremsstrahlung from a gas of temperature ≈ 8 keV. Which is the mass of the emitting gas? How much it contributes to the total mass of the cluster? Is it enough to explain the measured velocity dispersion of galaxies? Comment the result.

11 Inverse Compton scattering in the cosmic microwave background radiation

Relativistic electrons in the Galaxy lose energy due to inverse Compton scattering with photons in the Cosmic Microwave Background (CMB) radiation. The CMB is a black body emission of temperature $T = 3$ K and energy density ω_{CMB} ~0.25 eV/cm^3. We saw during the class that in an inverse Compton scattering the average energy of the up-scattered (high energy) photon is:

$$E_\gamma = \frac{4}{3}\gamma^2\epsilon \tag{7}$$

where γ is the Lorentz factor of the incident electron, and ϵ is the energy of the soft (low energy) photon of the CMB. We also saw that in the Thompson regime, i.e. when $\gamma\epsilon < m_e c^2$, the cross section of the process is the Thompson one σ_T.

1. Compute the typical (average) energy of photons in the CMB as $\langle\epsilon\rangle \approx kT$, and the corresponding energy of the up-scattered photons for energies of the incident electron equal to 1, 10, and 100 TeV. Above which energy of the electron the interaction is no longer in the Thomson regime?

2. Consider an electron moving through the CMB radiation with velocity c. Compute the number of scatterings per second it is experiencing (assume to be in the Thomson regime), as a function of ω_{CMB} and $\langle\epsilon\rangle$.

3. Using the result from the previous question, determine the expression for the power emitted by the electron due to inverse Compton scattering.

12 Inverse Compton scattering versus synchrotron emission

In the previous exercise, you showed that the power emitted by an ultra-relativistic electron due to inverse Compton scattering in the Thomson regime is:

$$P_{IC} \equiv -\frac{\mathrm{d}E}{\mathrm{d}t}|_{IC} = \frac{4}{3}\sigma_T c\gamma^2\omega_{CMB} \tag{8}$$

which is also equal to the rate of energy lost by the electron. This expression is very similar to that we obtained for synchrotron radiation:

$$P_{syn} \equiv -\frac{\mathrm{d}E}{\mathrm{d}t}|_{syn} = \frac{4}{3}\sigma_T c\gamma^2\omega_B \tag{9}$$

where $\omega_B = B^2/8\pi$ is the energy density of the ambient magnetic field.

Compute an effective magnetic field B_{eff} defined by imposing $P_{IC} = P_{syn}$. What is the physical meaning of such a field? How does it compare with the typical field found in the interstellar medium ($B \sim 3\mu$G)?

13 Inverse Compton versus proton-proton interactions

Consider a source (for example a supernova remnant shell characterised by a gas density $n \sim 4$ cm^{-3}) containing cosmic ray protons and electrons having an identical spectral energy distribution. Cosmic ray protons have a spectrum $N_p(E) = N_0^p(E/\text{TeV})^{-2.4}$ in the energy range $E > 1$ GeV, while electrons have a spectrum $N_e(E) = N_0^e(E/\text{TeV})^{-2.4}$ in the energy range $E > 0.5$ MeV (the rest mass energy of an electron $m_e c^2$). The spectra are normalised so that the sources contains an energy W_{CR}^p in cosmic ray protons and W_{CR}^e in cosmic ray electrons.

Cosmic ray protons produce gamma rays due to proton-proton interactions in the ambient gas, and the resulting luminosity is $Q_\gamma^p(E_\gamma)E_\gamma^2$. Cosmic ray electrons produce gamma rays due to inverse Compton scattering in the CMB radiation, and the resulting luminosity is $Q_\gamma^e(E_\gamma)E_\gamma^2$.

Compute the ratio W_{CR}^e/W_{CR}^p that would satisfy the condition: $Q_\gamma^p(E_\gamma)E_\gamma^2 = Q_\gamma^e(E_\gamma)E_\gamma^2$ at $E_\gamma = 1$ TeV.

Comment the result.

14 The diffusion equation

The equation describing the diffusive transport of cosmic rays of a given energy along magnetic field lines is:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right) + Q_0\delta(t)\delta(z) \tag{10}$$

where the injection term describes an impulsive injection of Q_0 particles at time $t = 0$ and at $z = 0$. Here, t is the time coordinate, and the field line is assumed to be aligned along the spatial coordinate z. The solution of this equation is:

$$N(t, z) = \frac{Q_0}{\sqrt{4\pi Dt}} \exp\left[-\frac{z^2}{4Dt}\right] \tag{11}$$

Compute the evolution with time of the mean displacement $\langle z \rangle$ of cosmic rays, and of the mean squared displacement $\langle z^2 \rangle$. What does $\langle z^2 \rangle$ represents? Can we use this result to estimate the typical residence time of comic rays in the region $|z| < L$?

[Hint: $\int_{-\infty}^{+\infty} \mathrm{d}x x^2 e^{-x^2} = \sqrt{\pi}/2$].

15 Characteristic energy loss time

Cosmic ray electrons of energy E loose energy through synchrotron radiation and/or inverse Compton scattering as:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -AE^2 \tag{12}$$

where A is a positive constant. Solve the equation above and found $E(t)$ assuming that the particle is characterised by an initial energy E_0 at $t = 0$. By using this result show that the expression $E/|\mathrm{d}E/\mathrm{d}t|$ computed at $E = E_0$ is a good estimate of the energy loss time of the particle.

What happens for an energy loss mechanism characterised by $\frac{\mathrm{d}E}{\mathrm{d}t} = -BE$ where B is a positive constant? Show that proton-proton interaction losses are described by this law.